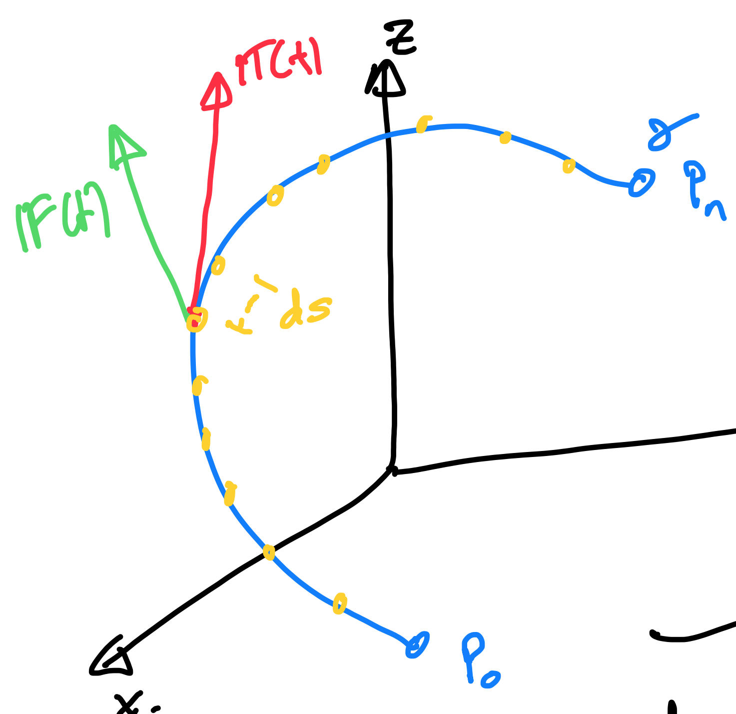
Line integrals in vector fields.

- As we showed before, a path integral inside a scalar potential field is:

$$\int_\gamma f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

- Now, suppose that we want to find the work done on a particle travelling inside a force field (vector field):



→ What we're doing is dividing the curve into small subsets. If we imagine a particle travelling on γ, then the only component of $\mathbf{F}(t)$ that does work on the particle is the component that is tangent to γ.

Here,
$$\text{Work} = \int_\gamma \mathbb{F} \cdot \mathbb{T}(s)\,ds$$

* Recall that: $\mathbb{T}(s) = \dfrac{d\mathbb{r}}{ds}$ *

Now, we can write out our integral as:

$$\boxed{\text{Work} = \int_\gamma \mathbb{F} \cdot \mathbb{T}\,ds = \int_\gamma \mathbb{F} \cdot d\mathbb{r} = \int_a^b \mathbb{F}(\mathbb{r}(t)) \cdot \mathbb{r}'(t)\,dt.}$$

- We've also shown that if a vector field is conservative, then:

$$\mathbb{F}(x,y,z) = \nabla\phi,$$ where ϕ is a scalar potential function.

→ Our expression for "work", for these cases, looks like:

$$\int_\gamma \mathbb{F}(x,y,z) \cdot d\mathbb{r} = \int_\gamma \nabla\phi \cdot d\mathbb{r}$$
$$= \int_a^b \nabla\phi(\mathbb{r}(t)) \cdot \mathbb{r}'(t)\,dt.$$

$$= \int_a^b \left[\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right] dt$$

$$= \int_a^b \partial\phi = \boxed{\phi(r(b)) - \phi(r(a))}$$

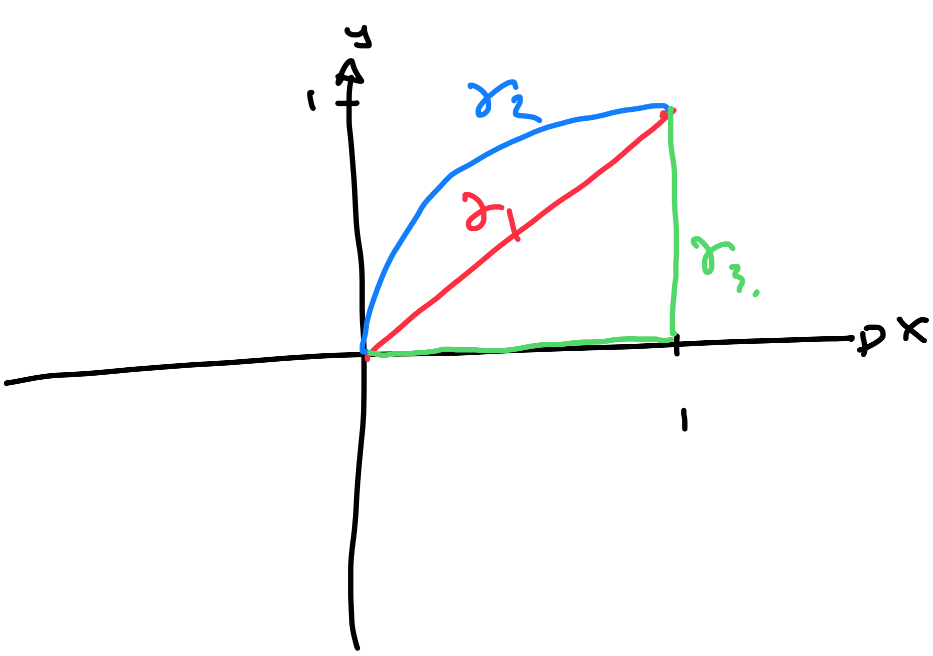
- Hence, for conservative vector fields, the "work" done on a particle travelling on some path γ_1 does not dependent on the path but only the endpoints.
  (More on this next class).

Ex.

$\mathbb{F}(x,y) = \langle xy, y^2+1 \rangle.$

$P_0 = (0,0)$

$P_1 = (1,1).$



$\gamma_1 := \{ y = x \}$

$\gamma_2 := \{ x = y^2 \}$

$$\gamma_3 := \begin{cases} y = 0,\ x = x & ;\ 0 \le x \le 1 \\ y = y,\ x = 1, & ;\ 0 \le y \le 1 \end{cases}$$

First, lets check to see if the vector function is conservative:

$$\nabla \times \mathbb{F} = \frac{\partial}{\partial x}(y^2+1) - \frac{\partial}{\partial y}(xy) = x \quad \therefore \text{ Not conservative}$$

Line γ_1

$y = x \Rightarrow x(t) = t.$

Hence, $\mathbf{r}(t) = \langle t, t \rangle.$

$\mathbf{r}'(t) = \langle 1, 1 \rangle.$

$$\therefore \int_{\gamma_1} \mathbb{F} \cdot d\mathbf{r} = \int_0^1 \langle t^2, t^2+1 \rangle \cdot \langle 1, 1 \rangle \, dt.$$

$$= \int_0^1 2t^2+1 \, dt = 5/3.$$

Line γ_2.

$y^2 = x \Rightarrow y(t) = t.$

$\mathbf{r}(t) = \langle t^2, t \rangle.$

$\mathbf{r}'(t) = \langle 2t, 1 \rangle.$

$$\therefore \int_{\gamma} \mathbb{F} \cdot d\mathbf{r} = \int_0^1 \langle t^3, t^2+1 \rangle \cdot \langle 2t, 1 \rangle \, dt.$$

$$= \int_0^1 2t^4 + t^2+1 \, dt = \frac{26}{15}$$

Line γ_3.

We need to break this into two sections.

Section #1 $\Rightarrow$ $y=0$, $x=t$.

$\mathbf{r}(t) = \langle t, 0 \rangle$,

$\mathbf{r}'(t) = \langle 1, 0 \rangle$. $t \in [0,1]$.

Section #2 $\Rightarrow$ $x=1$, $y=t$.

$\mathbf{r}(t) = \langle 1, t \rangle$.

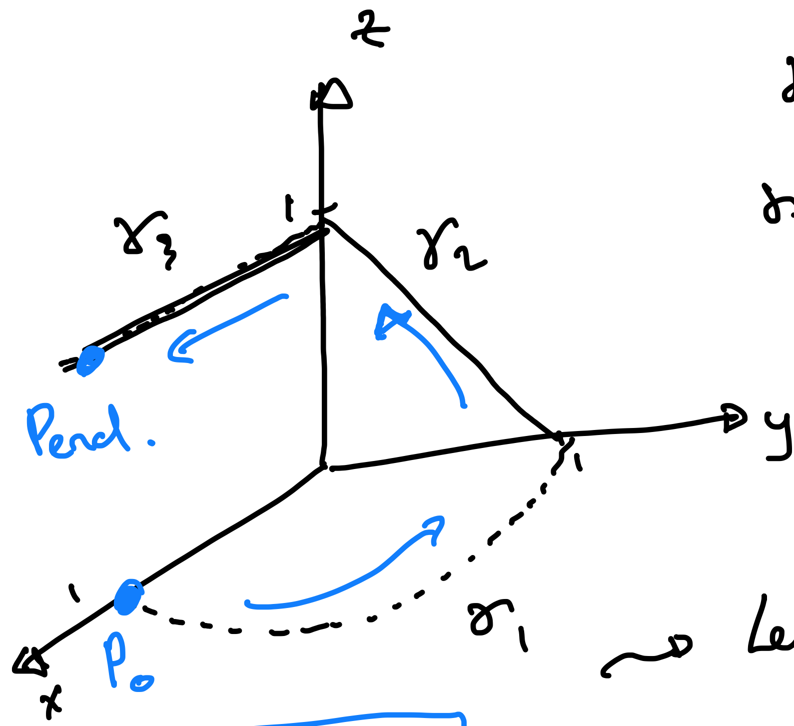
$\mathbf{r}'(t) = \langle 0, 1 \rangle$. $t \in [0,1]$.

Hence,

$$\int_{\gamma_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 0,0 \rangle \cdot \langle 1,0 \rangle \, dt + \int_0^1 \langle t, t^2+1 \rangle \cdot \langle 0,1 \rangle \, dt$$

$$= 0 + \int_0^1 t^2+1 \, dt = 4/3$$

Ex.



$\gamma_1: \mathbf{r}_1(t) = \langle \cos t, \sin t, 0\rangle,\ t \in [0, \pi/2]$

$\gamma_2: \mathbf{r}_2(t) = \langle 0,1,0\rangle + t\langle 0,-1,1\rangle,\ t \in [0,1]$

$\gamma_3: \mathbf{r}_3(t) = \langle 0,0,1\rangle + t\langle 1,0,0\rangle,\ t \in [0,1]$.

ie. $\boxed{\gamma = \gamma_1 + \gamma_2 + \gamma_3}$

$\rightsquigarrow$ Let $\mathbb{F}(x,y,z) = \langle y+z, x, x\rangle$.

$\rightarrow$ First, check if the vector function $\mathbb{F}(x,y,z)$ is <u>conservative</u>

$$\nabla \times \mathbb{F} = \left[\frac{\partial x}{\partial y} - \frac{\partial x}{\partial z}\right]\hat{i} - \left[\frac{\partial x}{\partial x} - \frac{\partial}{\partial z}(y+z)\right]\hat{j} + \left[\frac{\partial x}{\partial x} - \frac{\partial}{\partial y}(y+z)\right]\hat{k}$$

$$= 0$$

Hence
$$\int_\gamma \mathbb{F}(x,y,z)\cdot d\mathbf{r} = \phi(1,0,1) - \phi(1,0,0).$$

Let's first find ϕ...

$$\frac{d\phi}{\partial x} = y+z \Rightarrow \phi(x,y,z) = x(y+z) + f(y,z).$$

$$\frac{\partial \phi}{\partial y} = x \Rightarrow \phi(x,y,z) = xy + f(x,z).$$

$$\frac{\partial \phi}{\partial z} = x \Rightarrow \phi(x,y,z) = xz + f(x,y).$$

Hence $\phi(x,y,z) = x(y+z)$.

$$\therefore \boxed{\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \phi(1,0,1) - \phi(1,0,0) = 1}$$