Path Independence.

- Up to now we've tested "conservativeness" of vector fields by two methods:

1) $\mathbb{F}(x, y, z)$ is said to be conservative if there exists a scalar and continuous potential function such that: $\boxed{\mathbb{F} = \nabla \phi}$

2) $\mathbb{F}(x, y, z)$ is said to be conservative if the curl of the vector field is zero (i.e. the vector field is irrotational):

$$\boxed{\nabla \times \mathbb{F} = 0}$$

Let's show an example where we validate one method, but not the other.

Ex

$$\mathbb{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

[defined for all (x, y) in $\mathbb{R}^2$ except $(x, y) = (0, 0)$]

- If we test method #2, we can see that

$\nabla \times \mathbb{F} = 0$.

I'll let you all find this, but we end up with:

$$\phi(x,y) = \tan^{-1}(y/x)$$

- Now, we see the problem.... We showed that $\nabla \times \mathbb{F} = 0$, but the potential function $\phi(x,y)$ is really an angle $\theta(x,y)$ where $\theta(x,y) = \tan^{-1}(y/x) = \phi$, having infinite possibilities (i.e. infinite combination of y and x give the same angle $\theta(x,y) = \phi(x,y)$).

Really what this infers is that your potential function <u>must</u> be continuous & differentiable.

→ Recall that for conservative vector fields:

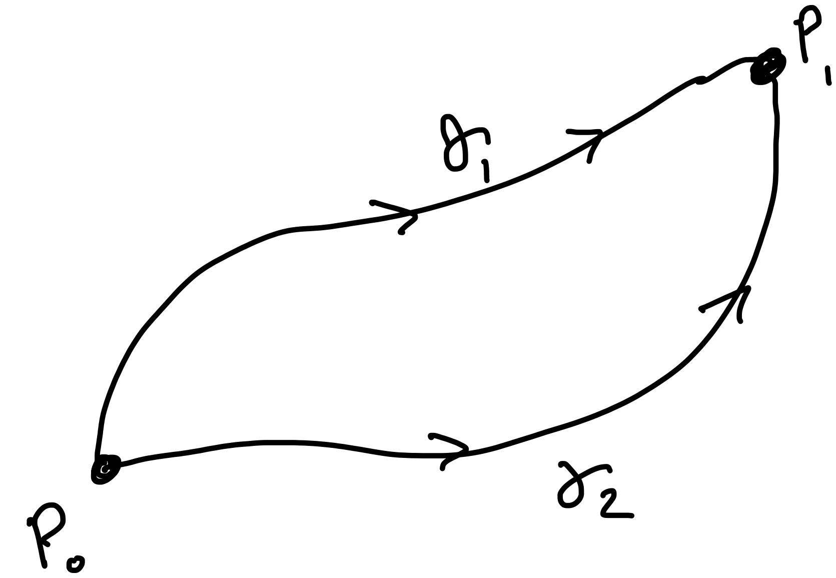
$$\int \mathbb{F} \cdot d\mathbf{r} = \phi(P_1) - \phi(P_0).$$

Suggesting that the path doesn't matter!

Ex.

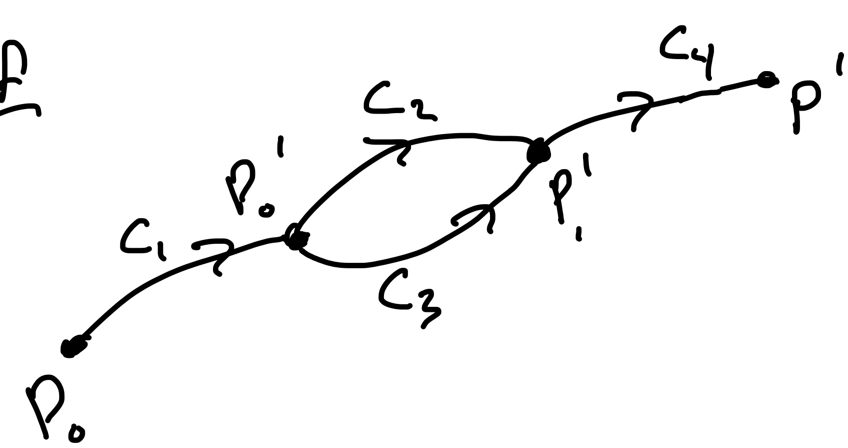
$$\int_{\gamma_1} \mathbb{F} \cdot d\mathbf{r} = \int_{\gamma_2} \mathbb{F} \cdot d\mathbf{r}$$

where:



- This is valid for any path from P_0 to P_1.

Proof



γ_1 : C_1, C_2, C_4

γ_2 : C_1, C_3, C_4

$$\int_{\gamma_1} \mathbb{F} \cdot d\mathbf{r} = \int_{\gamma_2} \mathbb{F} \cdot d\mathbf{r}$$

$$\int_{C_1} \mathbb{F} \cdot d\mathbf{r} + \int_{C_2} \mathbb{F} \cdot d\mathbf{r} + \int_{C_4} \mathbb{F} \cdot d\mathbf{r} = \int_{C_1} \mathbb{F} \cdot \partial\mathbf{r} + \int_{C_3} \mathbb{F} \cdot \partial\mathbf{r} + \int_{C_4} \mathbb{F} \cdot \partial\mathbf{r}$$

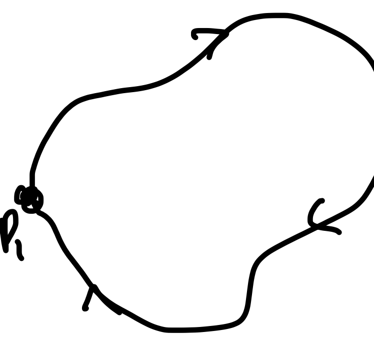
Hence $\int_{C_2} \mathbb{F} \cdot \partial\mathbf{r} = \int_{C_3} \mathbb{F} \cdot \partial\mathbf{r}$.

Take home message for today:

- For a continuous vector fields in $\mathbb{R}^2$ or $\mathbb{R}^3$, then:

① $\mathbb{F} = \nabla \phi$ if $\mathbb{F}$ is conservative.

② $\int_\gamma \mathbb{F} \cdot d\mathbf{r} = 0$ for "closed" curves, i.e. $P_0 = P_1$



③ The integral is path independent for curves that start and end at the same point

④ If $\mathbb{F}$ is continuous & differentiable on <u>all</u> $\mathbb{R}^2 / \mathbb{R}^3$, then $\mathbb{F}$ is conservative <u>if and only if</u> $\nabla \times \mathbb{F} = 0$.