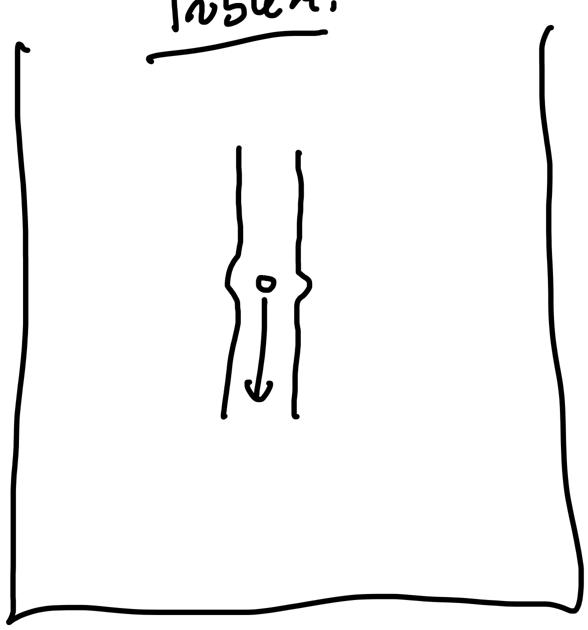
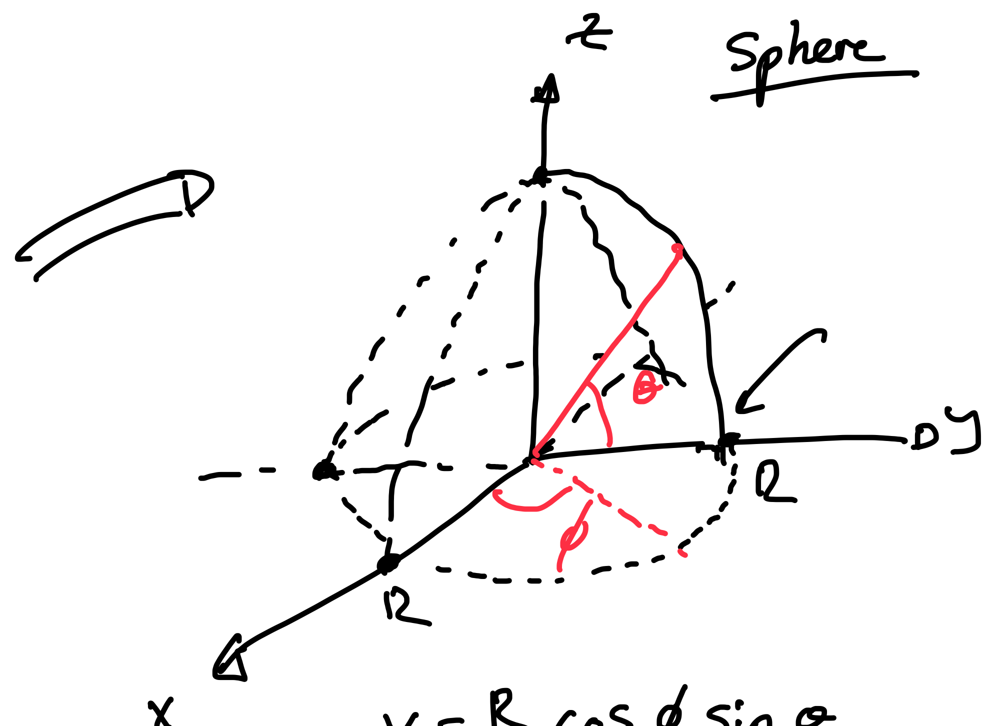
A classic problem.

- If you're interested, here is the solution to the classic "Stokes flow" problem of a falling sphere in a viscous fluid. Here, I'll derive the drag (viscous) forces acting on the sphere by use of a stream function.

Problem.



Sphere



$$x = R\cos\phi\sin\theta$$

$$y = R\sin\phi\sin\theta$$

$$z = R\cos\theta$$

Equations of motion

① $\nabla \cdot \vec{v} = 0$

② $\nabla^2 \vec{\omega} = 0$

AND,

$\vec{\omega} = \nabla \times \vec{v}$ (vorticity)

Stream function $\Rightarrow$ $\vec{v} = \nabla \times \psi$

Hence,

① $\nabla \cdot (\nabla \times \psi) = 0$

② $\nabla \cdot \nabla (\nabla \times (\nabla \times \psi)) = 0 = \nabla^4 \psi$

Divergence of $\vec{v}$ in spherical coordinates reads,

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} v_\phi}_{\text{(symmetry)}}^{\,0} = 0$$

Hence, we seek a stream function that satisfies:

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \quad ; \quad v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

→ Now, we can calculate the vorticity field..

we know that $\omega_r = \omega_\theta = 0$, hence,

$$\omega_\phi = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{-1}{\sin\theta}\frac{\partial \psi}{\partial r}\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}\right)\right]$$

$$= \frac{-1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial \theta^2} - \frac{1}{r^2 \sin\theta \tan\theta}\frac{\partial \psi}{\partial \theta}\right]$$

$$= \frac{-1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{r^2\tan\theta}\frac{\partial\psi}{\partial\theta}\right]$$

Now, we need to find:

$$\nabla^2 \vec{\omega} = \langle 0, 0, \nabla^2 \omega_\phi - \frac{\omega_\phi}{r^2 \sin\theta} \rangle.$$

$$= \nabla^2 \omega_\phi - \frac{\omega_\phi}{r^2\sin\theta}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \omega_\phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \omega_\phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\overbrace{\frac{\partial^2 \omega_\phi}{\partial \phi^2}}^{0} - \frac{\omega_\phi}{r^2 \sin^2\theta}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial w_\phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial w_\phi}{\partial \theta}\right) - \frac{w_\phi}{r^2 \sin^2\theta}$$

- Now, we sub in w_ϕ, but it looks nasty so let's sub in : $\frac{F}{r\sin\theta} = w_\phi$

Here,

$$\nabla^2 \vec{w} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{F}{r\sin\theta}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{F}{r\sin\theta}\right)\right] - \frac{F}{r^3\sin^3\theta}$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left[r^2\left(\frac{rF_r' - F}{r^2}\right)\right] + \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\sin\theta F_\theta' - F\cos\theta}{\sin^2\theta}\right)\right] - \frac{F}{r^3\sin^3\theta}$$

$$= \frac{1}{r^2\sin\theta}\left[rF_r'' + \cancel{F'} - \cancel{F'}\right] + \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left[F_\theta' - \frac{F}{\tan\theta}\right] - \frac{F}{r^3\sin^3\theta}$$

$$= \frac{1}{r^2 \sin\theta}\left(r F_r''\right) + \frac{1}{r^3 \sin\theta}\left[F_\theta'' - \frac{\tan\theta F_\theta' - \sec^2\theta F}{\tan^2\theta}\right] - \frac{F}{r^3 \sin^3\theta}$$

$$= \frac{1}{r^2 \sin\theta}\left(r \frac{\partial^2 F}{\partial r^2}\right) + \frac{1}{r^3 \sin\theta}\left[\frac{\partial^2 F}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial F}{\partial \theta} + \frac{1}{\sin^2\theta} F\right] - \frac{F}{r^3 \sin^3\theta}$$

$$= \frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\left[\frac{\partial^2 F}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial F}{\partial \theta}\right] = 0.$$

we essentially defined :

$$F = -\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{r^2 \tan\theta}\frac{\partial \psi}{\partial \theta}\right],$$

Hence, after some re-arranging, we get:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial}{\partial \theta}\right)\right]^2 \psi = 0.$$

Now, we use separation of variables...

$\psi = R\Theta$

$$\left.\begin{aligned} \psi_{rr} &= R''\Theta \\ \psi_{\theta\theta} &= R\Theta'' \\ \psi_{\theta} &= R\Theta' \end{aligned}\right\} \Rightarrow$$

sub into PDE yields

$$R''\Theta + \frac{1}{r^2}\left(R\Theta'' - \frac{1}{\tan\theta}R\Theta'\right) = 0.$$

$$(1) \quad \frac{R''}{R} = \frac{1}{r^2}\left(\frac{1}{\tan\theta}\frac{\Theta'}{\Theta} - \frac{\Theta''}{\Theta}\right) = \lambda$$

$\underline{\Theta}$

$$\Theta'' = \frac{\Theta'}{\tan\theta} - \Theta\lambda$$

Guess: $\Theta = \sin^2\theta$

$\Theta'' = 2\left[\cos^2\theta - \sin^2\theta\right]$

$\Theta' = 2\sin\theta\cos\theta$

$\therefore$ sub into (1) yields:

$$2\left[\cos^2\theta - \sin^2\theta\right] = \frac{2\sin\theta\cos\theta}{\tan\theta} - \sin^2\theta\,\lambda$$

$\therefore \boxed{\lambda = 2}$ $\underline{\text{AND}}$ $\Theta \propto \sin^2\theta$

Now, our stream function reads:

$$\psi = R\Theta = \sin^2\theta \, f(r).$$

Subing this into our PDE yields

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\underbrace{\left(\frac{\partial^2}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial}{\partial \theta}\right)}\right]^2 \sin^2\theta \, f(r) = 0$$

We already know that this is equal to

$$-2\Theta = -2\sin^2\theta$$

Hence, we can write:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}(-2)\right]^2 f(r) = 0.$$

A solution to the ODE takes the form:

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^3 + Er^4$$

Now, with $\psi = \sin^2\theta \, f(r)$;

$$v_r(r,\theta) = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta(r,\theta) = \frac{-1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

$$v_r(r,\theta) = \frac{2\cos\theta}{r^2}\left[\frac{A}{r} + Br + Cr^2 + Dr^3 + Er^4\right]$$

$$v_\theta(r,\theta) = \frac{-\sin\theta}{r}\left[-\frac{A}{r^2} + B + 2Cr + 3Dr^2 + 4Er^3\right]$$

Boundary Conditions.

① $v_r(\infty, 0) = v_c$ ← Far field velocity

② $v_r(R,\theta) = v_\theta(R,\theta) = 0$ ← No slip condition.

BC #1

$$v_c = 2\left[\frac{A}{r^3} + \frac{B}{r} + C + Dr + Er^2\right]$$

As $r \to \infty$, $D = E = 0$.

Hence, $v_c = 2C \to C = \frac{v_c}{2}$.

BC #2,

$v_r = 0 \quad 0 = \frac{A}{R} + BR + \frac{v_c R^2}{2} \Rightarrow A = -\frac{v_c R^3}{2} - BR^2$

$v_\theta = 0 \quad 0 = \frac{-A}{R^2} + B + v_c R \Rightarrow B = -\frac{3}{4} v_c R$

AND $A = \frac{v_c R^3}{4}$

Hence, we arrive at:

$$v_r(r,\theta) = \frac{2\cos\theta}{r^2}\left[\frac{U_c R^3}{4r} - \frac{3}{4}U_c R r + \frac{U_c r^2}{2}\right]$$

$$v_\theta(r,\theta) = \frac{-\sin\theta}{r}\left[\frac{-U_c R^3}{4r^2} - \frac{3}{4}U_c R + U_c r\right]$$

Now, we need to find the stresses acting on the sphere:

$$\sigma_z = \cos\theta\, \sigma_r - \sin\theta\, \sigma_\theta$$

Hence,

pressure

$$\tau_{rz} = \left[-P + 2\mu \frac{\partial v_r}{\partial r}\right]\cos\theta - \overset{\sin\theta}{}\left[\mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}^{\nearrow 0}\right]$$

viscosity of fluid

We recognize that:

$$\left.\frac{\partial v_r}{\partial r}\right|_{r=R} = 0 \quad ; \quad \left.\mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right|_{r=R} = \frac{-3\mu U_c \sin\theta}{2R}$$

$$\therefore \quad \tau_{rz} = -P\cos\theta + \frac{3\mu U_c \sin^2\theta}{2R}.$$

Now, we need to solve for the pressure.

$$\rightarrow \quad \frac{\partial P}{\partial r} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\tau_{\theta r}\sin\theta\right) - \frac{\tau_{\theta\theta}}{r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{rr}\right).$$

$$\tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

$$\tau_{\theta\theta} = 2\mu\left[\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right]$$

$$\tau_{rr} = 2\mu\frac{\partial v_r}{\partial r}.$$

Subing in, we get:

$$\frac{\partial P}{\partial r} = \frac{2U_c\mu\cos\theta}{r^2} - \frac{U_cR\mu}{r^3}\left[-4\left(\frac{R}{r}\right)^2 + 9\right]\cos\theta.$$

Integrating from $r \in [R, \infty]$

$$P = P_\infty - \frac{3U_c\mu}{2R}\cos\theta \quad \text{or} \quad \bar{P} = -\frac{3U_c\mu}{2R}\cos\theta.$$

Now, lets find the drag force (Finally!).

$$F_D = \iint R^2\,\tau_{rz}\sin\theta\,d\theta\,d\phi$$

$$= 2\pi R^2 \left[\int_0^{\pi} \frac{3}{2} \frac{U_c \mu}{R} \cos^2\theta \sin\theta \, d\theta + \frac{3}{2} \int_0^{\pi} \frac{\mu U_c \sin^3\theta}{R} \, d\theta \right]$$

$$= \boxed{6\pi\mu U_c R}$$

And there you have it, Stokes' law states that the drag force acting on a falling sphere under inertialess conditions is:

$$\boxed{F_D = 6\pi\mu U_c R}$$

Also, we can derive the drag coefficient as:

$$C_D = \frac{F_D}{\frac{1}{2}\rho U_c^2 (\text{Area})} = \frac{6\pi\mu U_c R}{\frac{1}{2}\rho U_c^2 (\pi R^2)} = \frac{12\mu}{\rho U_c R} = \frac{24}{Re_p}$$

where the Particle Reynolds number is:

$$Re_p = \frac{\rho U_c D_p}{\mu}$$

And $\rightarrow$ $\underline{D_p = 2R.}$