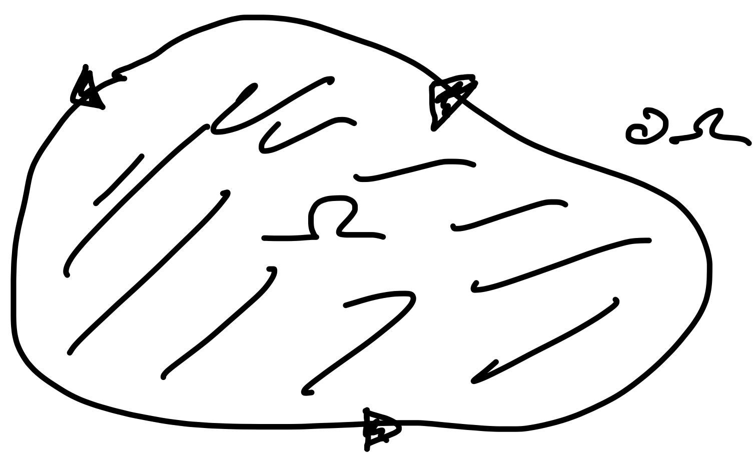
Green's Theorem Pt. II

- We can use Green's theorem to find the area encapsulating the line integral

ie.



Recall that:

$$\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} = \iint_{\Omega} \nabla \times \mathbb{F} \, dA.$$

Question?

What do we notice about this?

Well, if $\nabla \times \mathbb{F} = 1$, then:

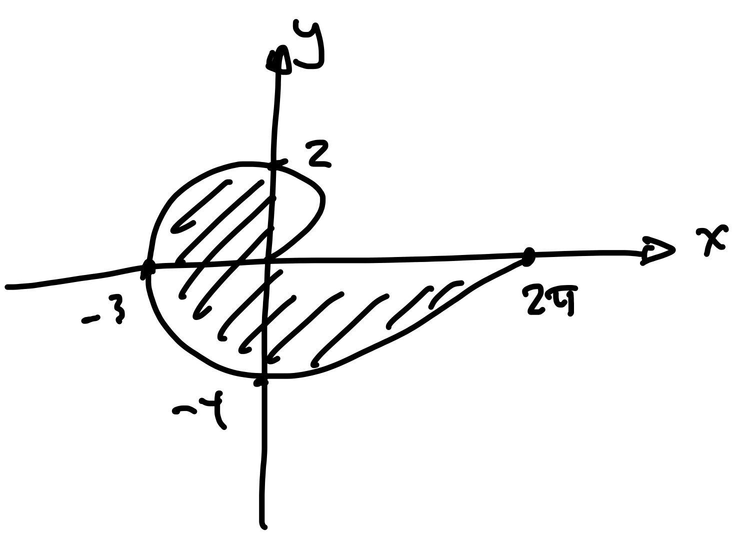
$$\boxed{\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} = \text{Area of } \Omega}$$

Essentially, we just made a pair of functions such that:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1$$

Consider the positively oriented spiral:

$r(t) = \langle t\cos t, t\sin t \rangle \qquad 0 \leq t \leq 2\pi$



← Find the highlighted area.

→ we need to find a vector function, such that:

$$\boxed{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1}$$

→ There are a few choices here

1) Set: $F_y = x, \quad F_x = 0.$

2) " : $F_x = -y, \quad F_y = 0.$

OR

3) Take the average of (1) & (2), i.e.

$$F_y = \frac{x}{2}, \quad F_x = -y/2. \qquad \text{(Preferred).}$$

Hence, $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{1}{2} + 1/2 = 1$

Now, let's write out the integral...

$$\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Omega} \nabla \times \mathbf{F} \, dA$$

$$\int_{\partial\Omega} \frac{-y}{2} \, dx + \frac{x}{2} \, dy = \iint_{\Omega} dA.$$

Now, given that we have:

$\mathbf{r}(t) = \langle t\cos t, t\sin t \rangle \quad 0 \le t \le 2\pi,$

$$\frac{1}{2}\int_0^{2\pi} x(t)\frac{dy(t)}{dt}\,dt - y(t)\frac{dx(t)}{dt}\,dt = \text{Area}$$

$$\frac{1}{2}\int_0^{2\pi} t^2(\cos^2 t + \sin^2 t)\,dt = \text{Area}.$$

Hence,

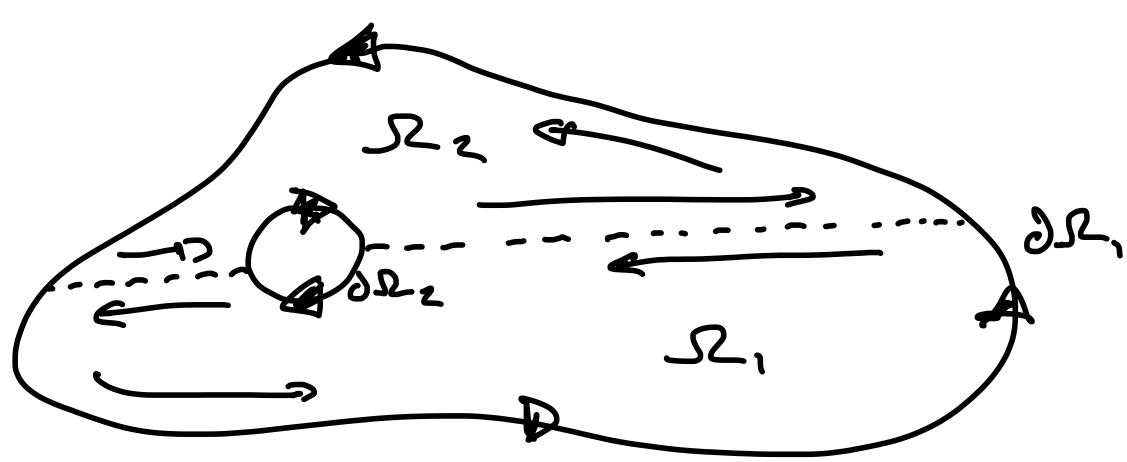
$$\text{Area} = \frac{1}{2}\int_0^{2\pi} t^2\,dt = \boxed{\frac{8\pi^3}{6}}$$

→ This leads us to another use of Green's theorem that surrounds vector fields that are $\underline{\underline{\text{not}}}$ continuous or differentiable at every point with the surface Ω, i.e. have holes in the region.

→ Imagine that we have a region with a hole in it, ie.

we can see that $\partial\Omega_1$ is positively oriented & $\partial\Omega_2$ is negatively oriented, hence, they oppose eachother.

With this, we can break Ω into two separate regions.



- What we see is that the line integrals along common boundaries cancel, yielding:

$$\iint_{\Omega} \nabla \times \mathbb{F} \, dA = \iint_{\Omega_1} \nabla \times \mathbb{F} \, dA + \iint_{\Omega_2} \nabla \times \mathbb{F} \, dA$$

which, can also be written as:

$$\iint_{\Omega} \nabla \times \mathbb{F} \, dA = \int_{\partial \Omega_1} \mathbb{F} \cdot d\mathbb{r} + \int_{\partial \Omega_2} \mathbb{F} \cdot d\mathbb{r} = \int_{\partial \Omega} \mathbb{F} \cdot d\mathbb{r}$$

$\therefore$ we arrive at.

$$\boxed{\iint_{\Omega} \nabla \times \mathbb{F} \, dA = \int_{\partial \Omega_1} \mathbb{F} \cdot d\mathbb{r} + \int_{\partial \Omega_2} \mathbb{F} \cdot d\mathbb{r} = \int_{\partial \Omega} \mathbb{F} \cdot d\mathbb{r}}$$

<u>Ex,</u>

Show that $\int_{\partial \Omega} \mathbb{F} \cdot d\mathbb{r} = 2\pi$ for <u>any</u> simple closed path that encloses the origin, for

$$\mathbb{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

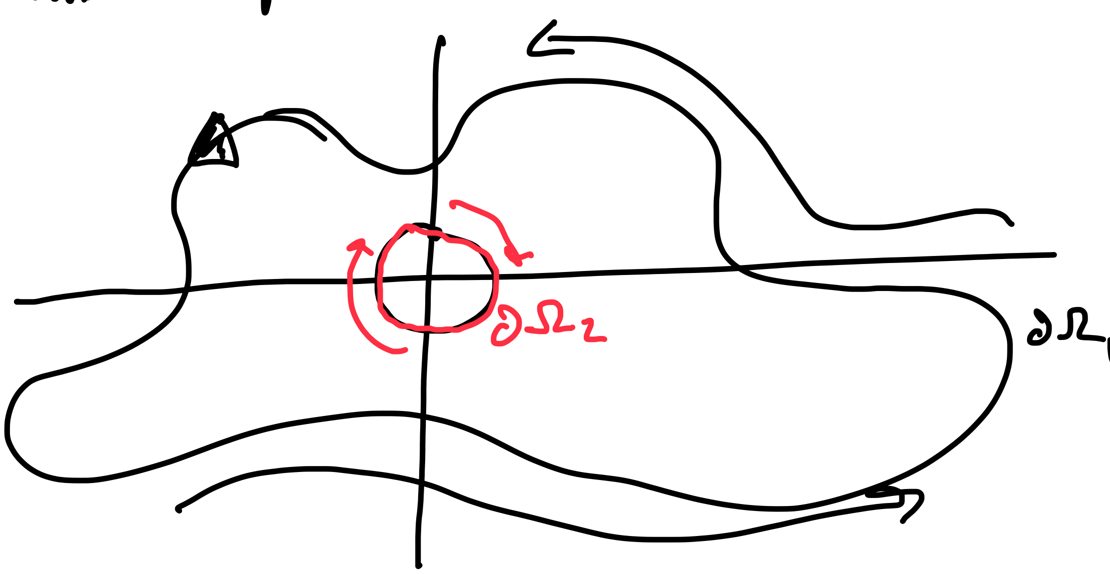
what we see is that $\mathbb{F}(0,0)$ is not defined, and nor is $\nabla \times \mathbb{F}$ at $(x,y) = (0,0)$....

but also,

$$\iint_{\Omega} \nabla \times \mathbb{F} = \iint_{\Omega} \left[\frac{x^2+y^2 - x(2x)}{(x^2+y^2)^2} - \frac{-(x^2+y^2) + 2y^2}{(x^2+y^2)^2} \right] dA.$$

$$= 0$$

Hence, this implies that :



$$\int_{\partial\Omega_1} \mathbb{F}\cdot d\mathbf{r} = \int_{\partial\Omega_2} \mathbb{F}\cdot d\mathbf{r} \quad \text{For } \underline{\underline{\text{any}}} \text{ closed curve } \partial\Omega_2.$$

Hence, if we let : $\partial\Omega_2 := \left\{ \begin{array}{l} \mathbf{r}(t) = \langle a\cos t, a\sin t \rangle, \\ 0 \le t \le 2\pi \end{array} \right\}$

then,

$$\int_{\partial\Omega_2} \mathbb{F}\cdot d\mathbf{r} = \int_0^{2\pi} \frac{a^2\sin^2 t + a^2\cos^2 t}{a^2\cos^2 t + a^2\sin^2 t}\, dt = \boxed{2\pi}$$