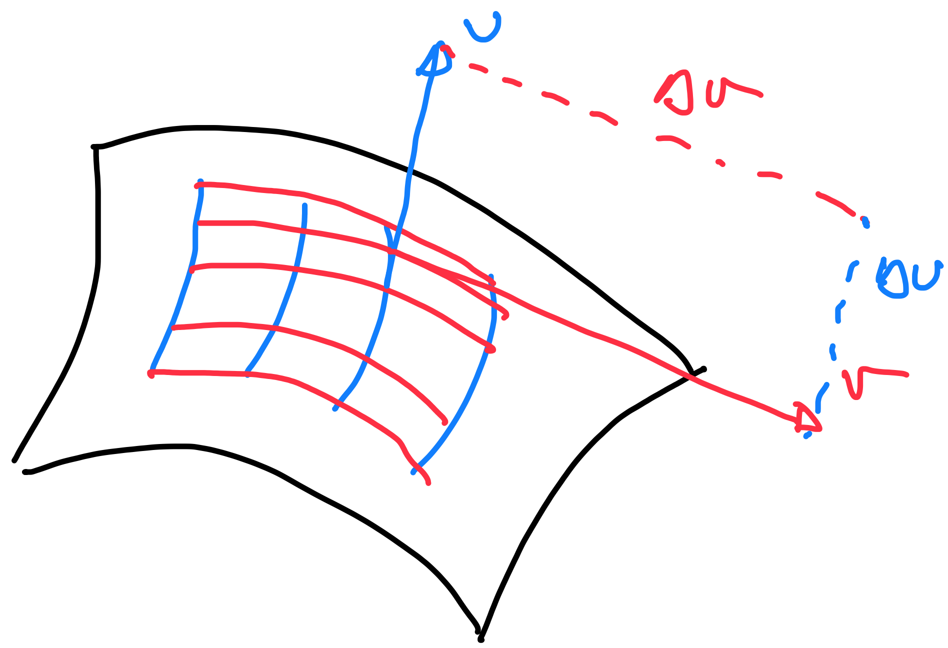
Surface Area.

- Suppose that you have a complex surface and you're interested in finding it's surface area.



We can construct a tangent plane at $\mathbf{r}(u_0, v_0)$ such that:

$$\mathbf{r}_u(u_0, v_0) = \mathbf{T}_u$$

AND

$$\mathbf{r}_v(u_0, v_0) = \mathbf{T}_v$$

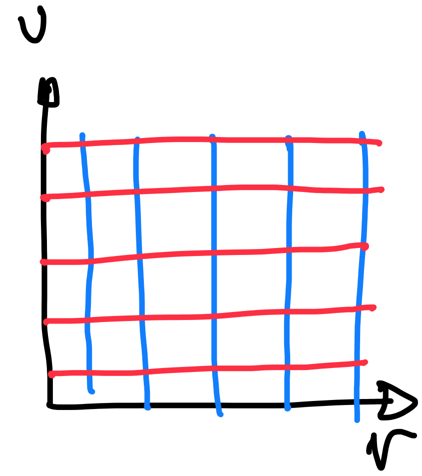
The height / width of the plane is simply:

$|\mathbf{r}_u(u,v)\, \Delta u| \rightarrow$ Height

$|\mathbf{r}_v(u,v)\, \Delta v| \rightarrow$ Width.

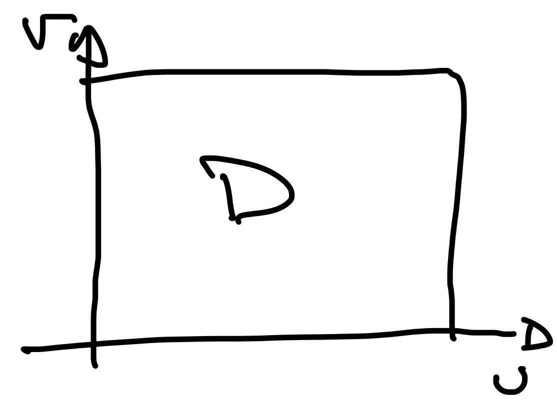
Hence, the area of the surface approximated by the tangent plane is:

$$\sum_{i=1}^{N} |\mathbb{r}_u \times \mathbb{r}_v| \Delta u_i \Delta v_i \quad , \text{ where } \Delta u_i = \frac{\Delta u}{N} \rightarrow$$



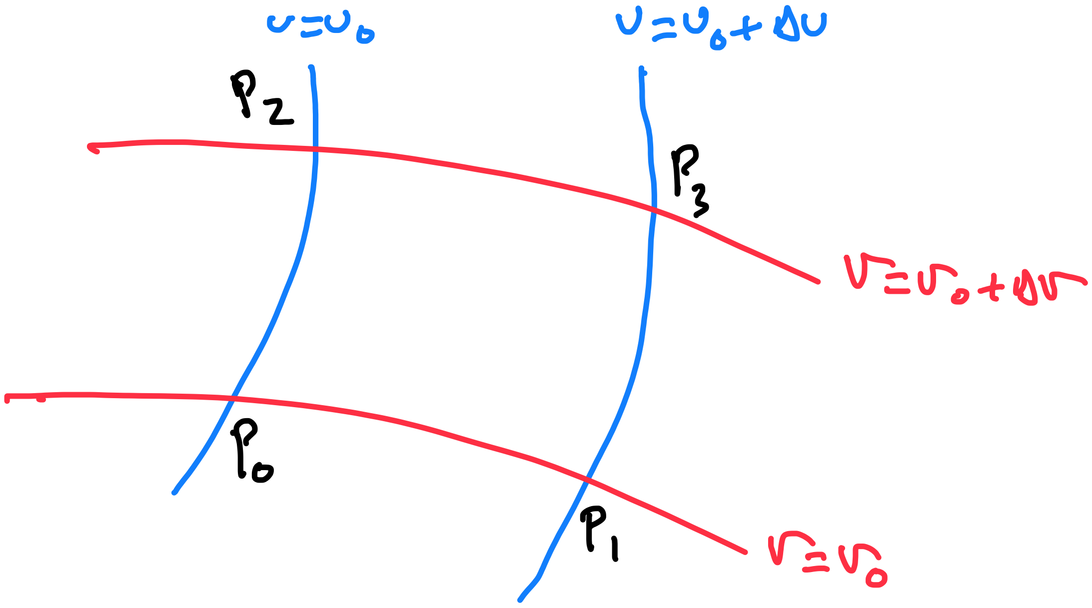
In the limit as Δu and $\Delta v \rightarrow 0$, and $N \rightarrow \infty$,

$$\iint_D |\mathbb{r}_u \times \mathbb{r}_v| \, dA =: \mathbb{A}(\Omega) \; , \text{ where 'D' is our parametrized domain.}$$



- A few things to note:

1) If we isolate a small region, i.e. $\Delta u / \Delta v$ are small, we can see that the surface can be linearly approximated:



Here, we can see that.

$$P_0 = \mathbf{r}(u_0, v_0) \qquad P_1 = \mathbf{r}(u_0+\Delta u, v_0)$$

$$P_2 = \mathbf{r}(u_0, v_0+\Delta v) \qquad P_3 = \mathbf{r}(u_0+\Delta u, v_0+\Delta v).$$

We can approximate these as!

$$P_0 \sim \mathbf{r}(u_0, v_0)$$

$$P_1 \sim \mathbf{r}(u_0, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\, \Delta u$$

$$P_2 \sim \mathbf{r}(u_0, v_0) + \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\, \Delta v$$

$$P_3 \sim \mathbf{r}(u_0, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\, \Delta u + \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\, \Delta v$$

Here,

$$\overrightarrow{P_0P_1} \approx \overrightarrow{P_2P_3} \approx \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\, \Delta u$$

$$\overrightarrow{P_0P_2} \approx \overrightarrow{P_1P_3} \approx \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\, \Delta v$$

The area of the cell is simply,

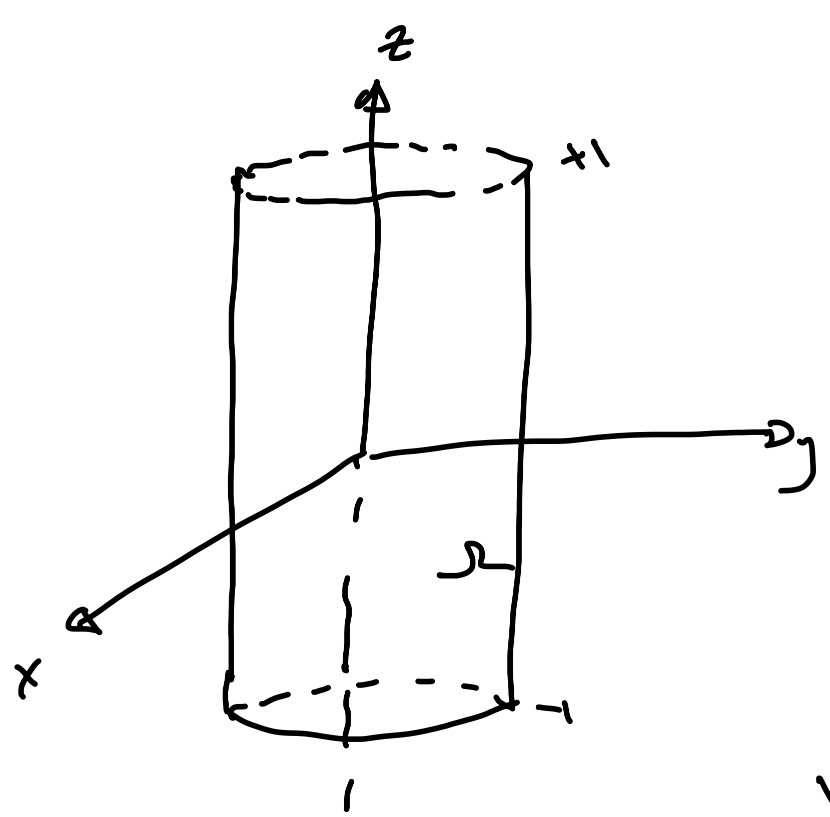
$$\underbrace{|\overrightarrow{P_0P_1}||\overrightarrow{P_1P_3}|\sin\theta}_{\text{Cross product!}} = |\overrightarrow{P_0P_1} \times \overrightarrow{P_1P_3}| \sim \left| \frac{\partial r}{\partial u}(u_0, v_0) \times \frac{\partial r}{\partial v}(u_0, v_0) \right| du\,dv$$

- The area of the cell is thus equivalent to to the magnitude of the vector that is orthogonal to the plane.

Ex.

Find the surface area of the cylinder.

$$\Omega = \{ x^2+y^2 = 4, \ -1 \le z \le 1 \}$$



Let's first parametrize:

$x = 2\cos u$

$y = 2\sin u$

$z = v$

Here, $r(u,v) = \langle 2\cos u, 2\sin u, v \rangle$.

u varies within $[0, 2\pi]$

v " " $[-1, 1]$

Here, $(u,v) \in [0, 2\pi] \times [-1, 1]$

$$\text{Area} = \iint_D \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA$$

$$\mathbf{r}_u(u,v) = \langle -2\sin u, 2\cos u, 0 \rangle.$$

$$\mathbf{r}_v(u,v) = \langle 0, 0, 1 \rangle.$$

Hence, $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \langle 2\cos u, 2\sin u, 0 \rangle.$

$\therefore$

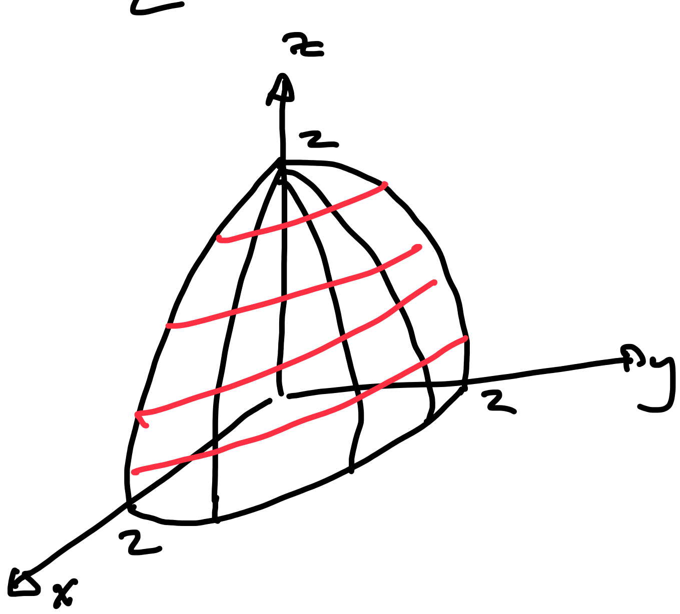
$$A(\Omega) = \int_0^{2\pi}\int_{-1}^{+1} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dA = \int_0^{2\pi}\int_{-1}^{1} 2\, dv\, du$$

$$= \underline{8\pi}$$

<u>Ex.</u>

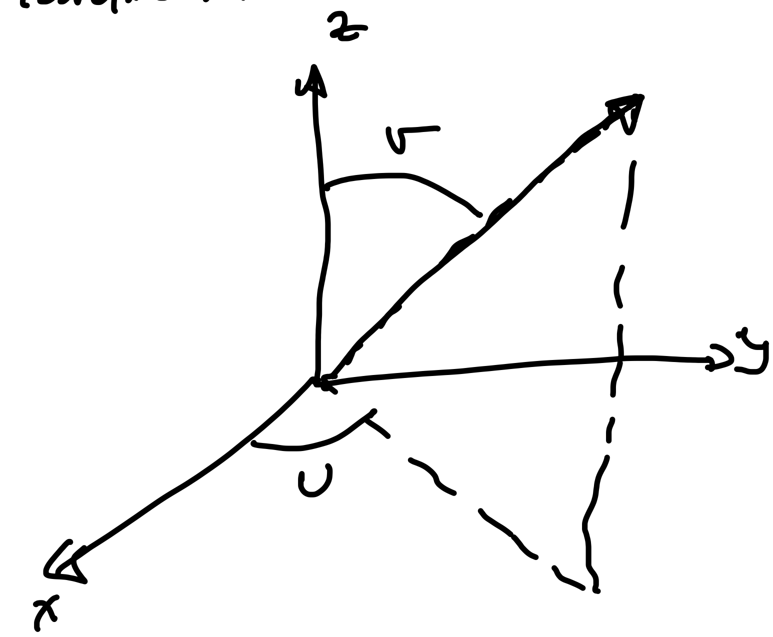
Surface area of a hemisphere.

$$\Omega = \left\{ x^2+y^2+z^2 = 4, \quad z \geq 0 \right\}$$



Parametrize with spherical coordinates.

$$x = 2\sin v \cos u$$
$$y = 2 \sin u \sin v$$
$$z = 2\cos v$$

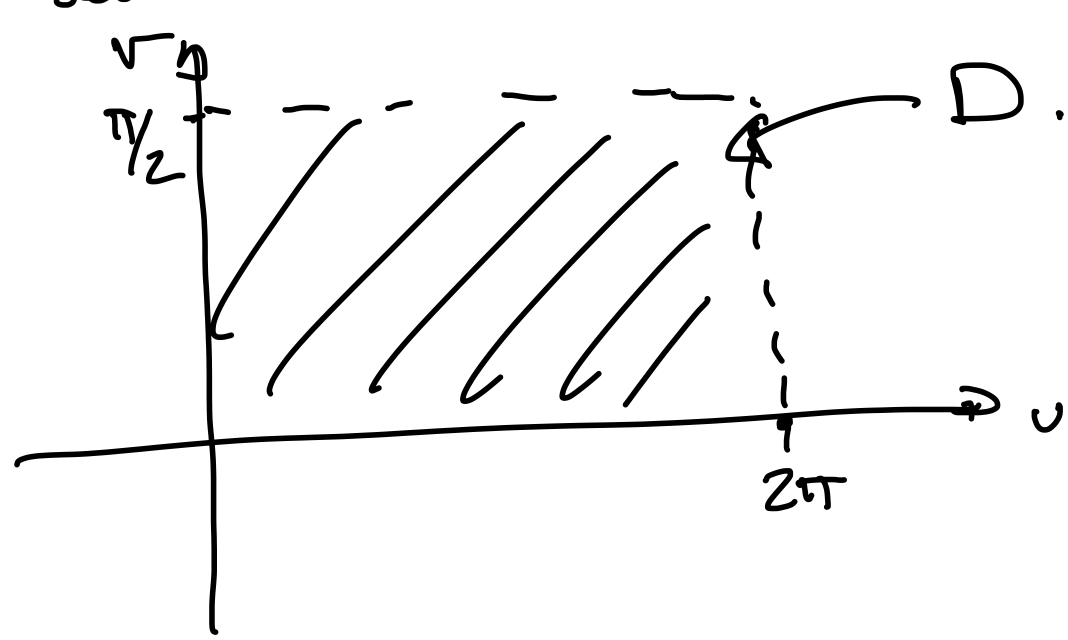


If we let $u \in [0, 2\pi]$ AND

$v \in [0, \pi/2]$, we

have our hemisphere..

Our parametrized domain looks like:



Here, $A(\Omega) = \iint_D \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du\,dv$

$$\frac{\partial r}{\partial u} = \langle -2\sin u \sin v,\ 2\cos u \sin v,\ 0 \rangle.$$

$$\frac{\partial r}{\partial v} = \langle 2\cos u \cos v,\ 2\sin u \cos v,\ -2\sin v \rangle$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \langle -4\cos u \sin^2 v,\ -4\sin u \sin^2 v,\ -4\sin^2 u \sin v \cos v - 4\cos^2 u \sin v \cos v \rangle$$

$$= -4 \langle \cos u \sin^2 v,\ \sin u \sin^2 v,\ \sin v \cos v \rangle.$$

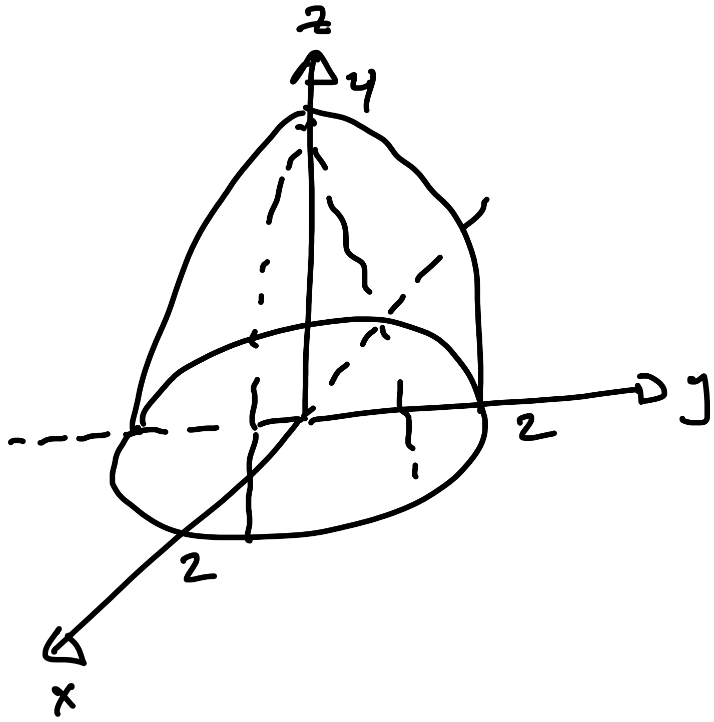
With a little re-arranging, we arrive at:

$$A(\Omega) = \int_0^{\pi/2}\int_0^{2\pi} 4\sin v\, du\,dv = 8\pi \int_0^{\pi/2} \sin v\, dv = 8\pi$$

Ex.

Find the surface area of the paraboloid that lies above the (x,y)-plane.

$$\Omega = \{ x^2+y^2+z=4,\quad 0 \le z \le 4 \}$$



let: $x = \sqrt{4-v}\cos u$

$y = \sqrt{4-v}\sin u$

$z = v$

hence,

$\mathbf{r}(u,v) = \langle \sqrt{4-v}\cos u, \sqrt{4-v}\sin u, v \rangle$

$\therefore \dfrac{d\mathbf{r}}{du} = \langle -\sqrt{4-v}\sin u, \sqrt{4-v}\cos u, 0 \rangle.$

$\dfrac{d\mathbf{r}}{dv} = \langle -\frac{1}{2}(4-v)^{-1/2}\cos u, -\frac{1}{2}(4-v)^{-1/2}\sin u, 1 \rangle.$

Hence, $\left| \dfrac{d\mathbf{r}}{du} \times \dfrac{d\mathbf{r}}{dv} \right| = \sqrt{17/4 - v}$

$$A(\Omega) = \int_0^4 \int^{2\pi} \left(\frac{17}{4} - v\right)^{1/2} du\, dv$$

$$= 2\pi \int_0^4 \left(\frac{17}{4} - v\right)^{1/2} dv$$

let $m = \frac{17}{4} - v$

$dm = -dv$

$$= -2\pi \int_{17/4}^{1/4} m^{1/2}\, dm$$

$$= \boxed{\frac{\pi}{6}\left[17^{3/2} - 1\right]}$$