More on parametrizing surfaces.

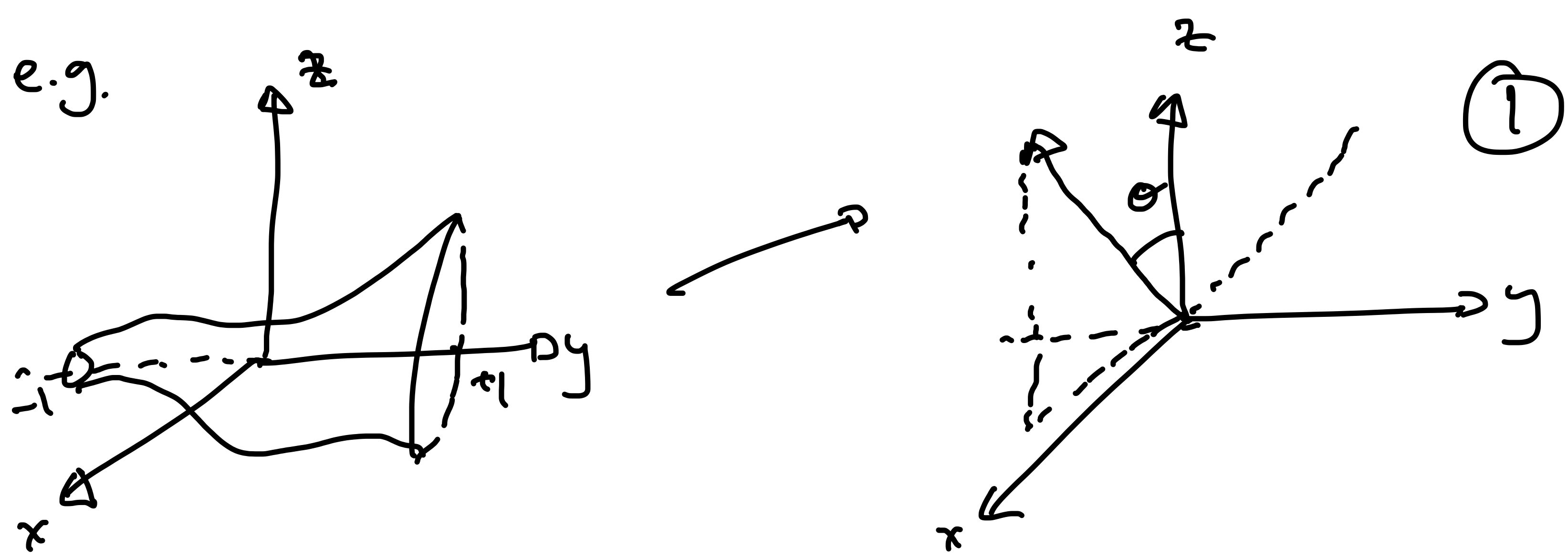
- I've shown in a few examples (ie. cones) that a useful parametrization is:

$\mathbf{r}(u,v) = \langle f(v)\cos u,\ f(v)\sin u,\ v \rangle$,

ie. a cylindrical coordinate system.

- This is useful when some dimension of a circular surface varies as a function of another.

e.g.

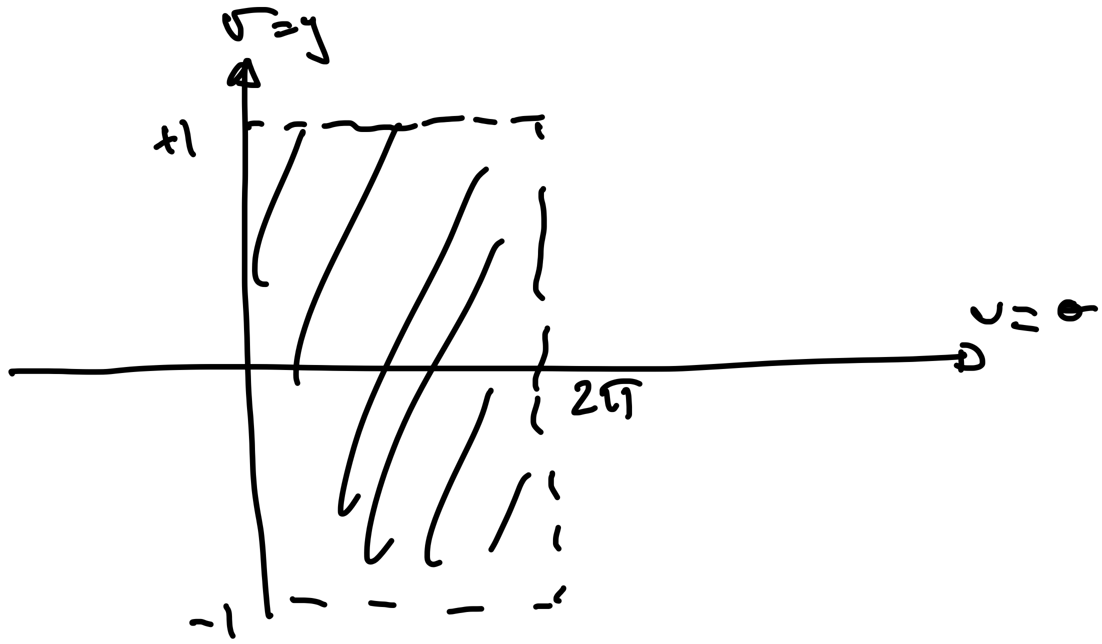


Here, a useful parametrization is:

$\mathbf{r}(u,v) = \langle f(v)\sin u,\ v,\ f(v)\cos u \rangle$

where $u = \theta$ in 1.

This ensures a rectangular parametrization domain.



Let's solve the same problem from last class using a cartesian parametrization.

Recall that: $\Omega = \left\{ x^2+y^2+z=4, \quad z \geq 0 \right\}$

let: $x = u$
$y = v$
$z = 4-u^2-v^2$

$\rightsquigarrow \mathbf{r}(u,v) = \langle u, v, 4-u^2-v^2 \rangle.$

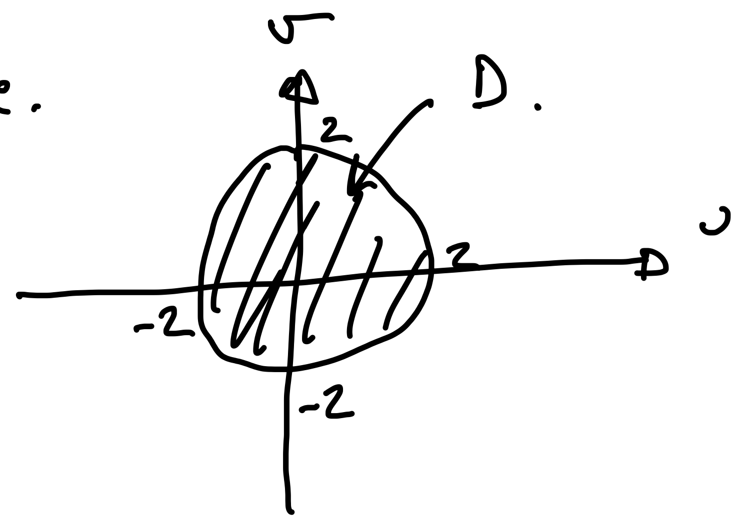
$\therefore \quad \dfrac{\partial \mathbf{r}}{\partial u} = \langle 1, 0, -2u \rangle$

$\dfrac{\partial \mathbf{r}}{\partial v} = \langle 0, 1, -2v \rangle.$

hence $\left| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right| = 2\sqrt{u^2+v^2+1/4}$

Now, what do we see?

Well, first of all our parametrized domain is circular, i.e.



Second, we notice that: $r^2 = u^2 + v^2$,

hence we can rewrite $|\mathbf{r}_u \times \mathbf{r}_v|$ as:

$$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = 2\sqrt{r^2 + 1/4}.$$

Now, we just need to change our ∂A term to:

$$\partial A = \partial u \partial v = r \partial r \partial \theta \leftarrow \text{Jacobian determinant.}$$

Here,

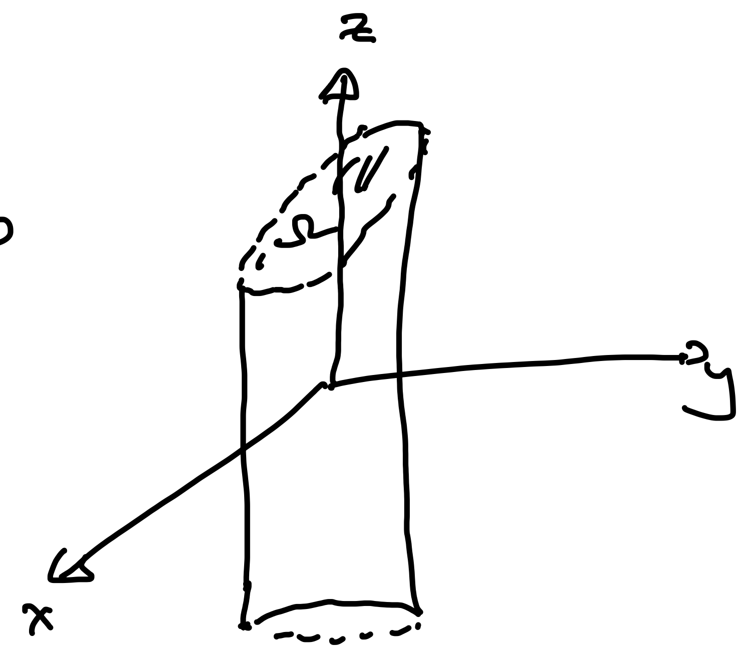
$$A(\Omega) = 2\int_0^{2\pi}\int_0^2 r\sqrt{r^2 + 1/4}\ \partial r \partial \theta$$

$$= 4\pi \int_0^2 r\sqrt{r^2 + 1/4}\ \partial r = \boxed{\frac{\pi}{6}\left[17^{3/2} - 1\right]}$$

Same as <u>before</u>.

Ex.

Find the area of the plane that cuts through a cylinder!



Plane: $x + 2y + 3z = 1$

cylinder: $x^2 + y^2 = 3$.

Here's a great example where a cartesian system should be used to parametrize:

$$x = u$$
$$y = v$$
$$z = \frac{1}{3}(1 - 2v - u)$$

$$\frac{\partial r}{\partial u} = \langle 1, 0, -1/3 \rangle$$

$$\frac{\partial r}{\partial v} = \langle 0, 1, -2/3 \rangle$$

Here, $\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| = \sqrt{1 + 1/9 + 4/9} = \frac{\sqrt{14}}{3}.$

$$A(\Omega) = \iint_D \frac{\sqrt{14}}{3}\, dA.$$

Recall that $D = \{ x^2 + y^2 \leq 3 \}$

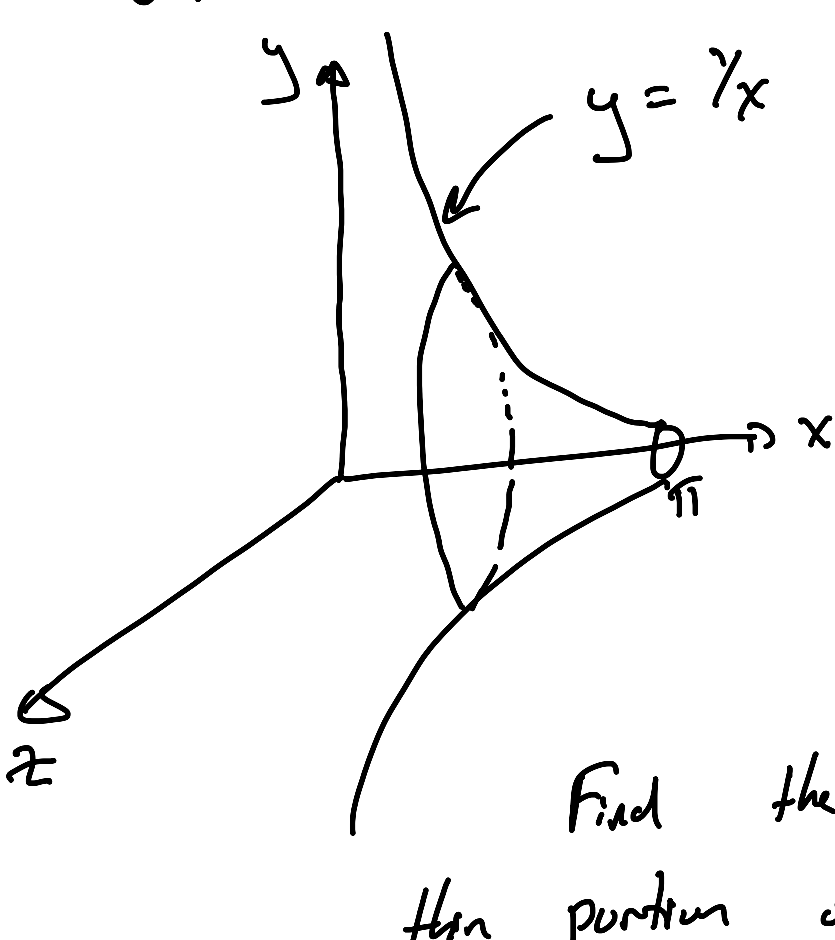
Hence

$$A(\Omega) = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{\sqrt{14}}{3}\, r\, dr\, d\theta$$

$$= \boxed{\pi\sqrt{14}}$$

Ex

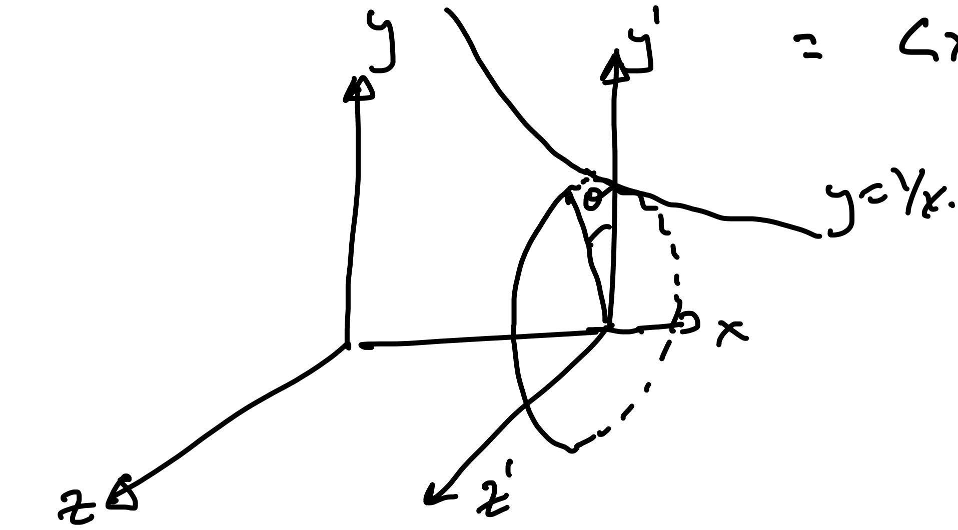
Gabriel's horn.



Find the surface area of a thin portion of Gabriel's horn.

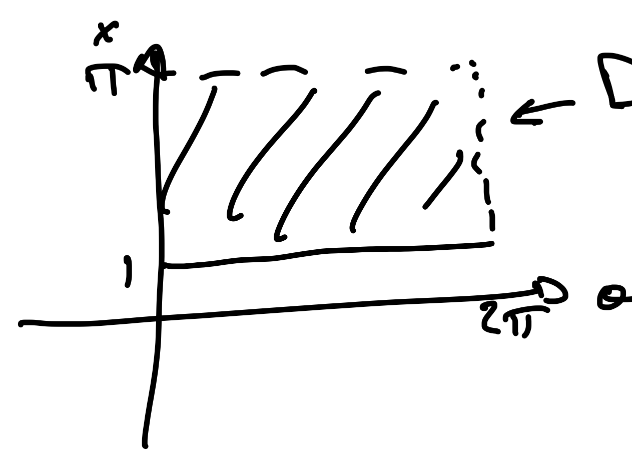
Parametrize as: $\mathbf{r}(x,\theta) = \langle x, r\cos\theta, r\sin\theta \rangle$.

$= \langle x, \frac{1}{x}\cos\theta, \frac{1}{x}\sin\theta \rangle$.



Let's solve for $1 \le x \le \pi$ AND $0 \le \theta \le 2\pi$.

Hence our domain is:



$$\frac{\partial \mathbf{r}}{\partial x} = \left\langle 1, -\frac{\cos\theta}{x^2}, -\frac{\sin\theta}{x^2} \right\rangle$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = \left\langle 0, -\frac{\sin\theta}{x}, \frac{\cos\theta}{x} \right\rangle.$$

Hence, $\left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| = \sqrt{x^{-6} + x^{-2}} = \left(\frac{x^4+1}{x^6} \right)^{1/2}$

$$\therefore \quad A(\Omega) = \iint_D \left(\frac{x^4+1}{x^6} \right)^{1/2} dx\, d\theta$$

let $u = x^{-4}$

$\partial u = -4x^{-5}\,\partial x.$

Here,

$$A(S) = \int_0^{2\pi}\int_1^{\pi^{-4}} \frac{-1}{4}\sqrt{u^{-2}+u^{-1}}\,\partial u\,\partial\theta$$

$$= \frac{-\pi}{2}\int_1^{\pi^{-4}} \sqrt{u^{-2}+u^{-1}}\,\partial u$$

$$= \frac{-\pi}{2}\int_1^{\pi^{-4}} \sqrt{\frac{u+1}{u}}\,\partial u$$

sub $m = \sqrt{u+1}$

$\partial m = \frac{1}{2}(u+1)^{-1/2}\,\partial u.$

$$= -\pi\int_{\sqrt{2}}^{\sqrt{\pi^{-4}+1}} \frac{m^2}{m^2-1}\,\partial m$$

$$= -\pi\left[m+\frac{1}{2}\left(\ln|1-m| - \ln(1+m)\right)\right]_{\sqrt{2}}^{\sqrt{\pi^{-4}+1}}$$

$$\approx \frac{5\pi}{2}$$