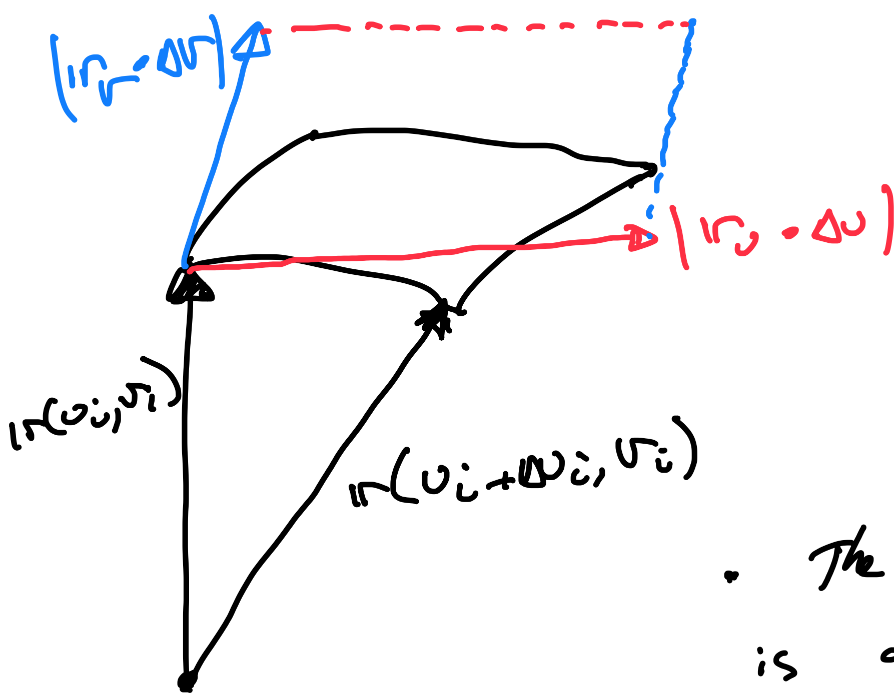
Surface Integrals Con't.

- Recall that we can approximate the area of a surface by constructing "pieces" of tangent planes.



- The area of the surface is approximated by the tangent plane, here.

$$\boxed{\text{Area} = \sum_{i=1}^{N} |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u_i \Delta v_i}$$

- We can take this a step further and weight each surface element by a scalar function "f", where f is continuous...

$$\underbrace{\sum_{i=1}^{N} f(\mathbf{r}(u_i, v_i))}_{\text{weight function.}} \underbrace{|\mathbf{r}_u(u_i, v_i) \times \mathbf{r}_v(u_i, v_i)| \, \Delta u_i \Delta v_i}_{\text{Area of a small tangent plane}}$$

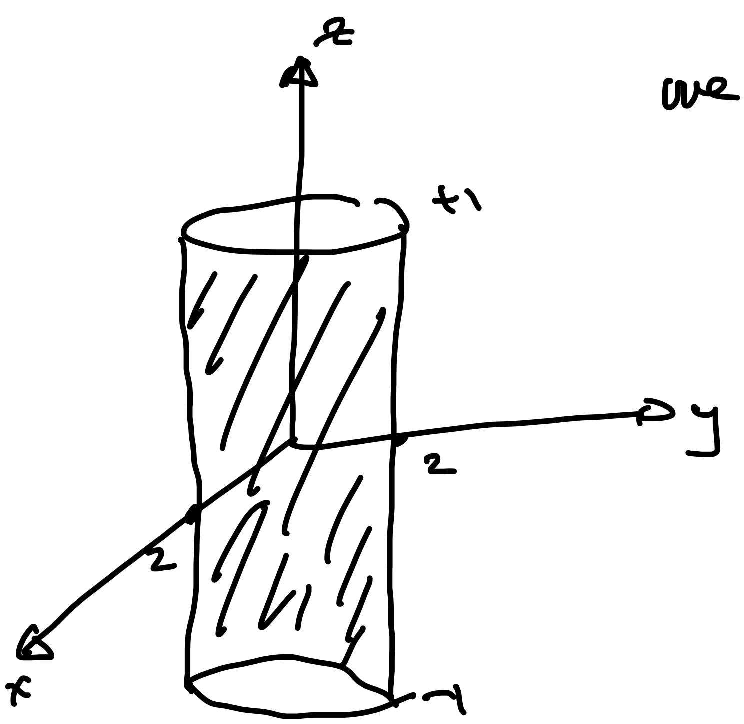
• This is a surface integral of a scalar function... ie.

$$\boxed{\iint_{\Omega} f(x,y,z)\, d\Omega = \iint_{D} f(\mathbf{r}(u,v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA}$$

Small piece of the surface

Ex.

let : $\Omega = \{ x^2+y^2=4 \ , \ -1 \leq z \leq 1 \}$ and



we seek to find

$$\iint_{\Omega} xy \, d\Omega.$$

① Parametrize the surface:

$x = 2\cos u$

$y = 2\sin u$

$z = v$

Hence, $\mathbf{r}_u(u,v) = \langle -2\sin u, 2\cos u, 0 \rangle$

$\mathbf{r}_v(u,v) = \langle 0, 0, 1 \rangle$.

Hence, $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4\cos^2 u + 4\sin^2 u} = 2.$

Our new parametrized domain D is:

$$D = \{ (u,v) \in [0, 2\pi] \times [-1, 1] \}$$

Our integral thus reads:

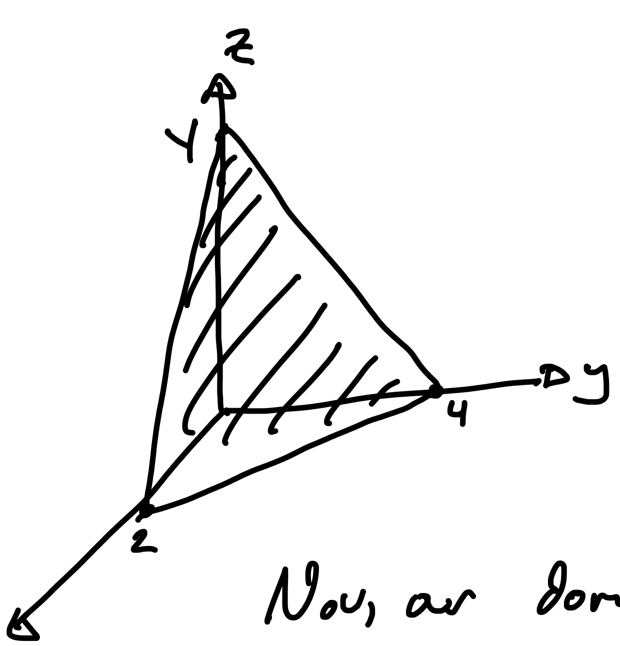
$$2\int_0^{2\pi}\int_{-1}^{1} 2\cos u \, 2\sin u \, dv\,du$$

$$= 4\int_0^{2\pi}\int_{-1}^{1} \sin(2u)\, dv\,du$$

$$= 4\int_0^{4\pi} \sin(m)\, dm = 0$$

<u>Ex.</u>

Find $\iint_{\Omega} x^2 \, d\Omega$ for $\Omega = \{ 2x+y+z = 4 \ , \ x \geq 0, \ y \geq 0, \ z \geq 0 \}$



$\longrightarrow$ Let's parametrize:

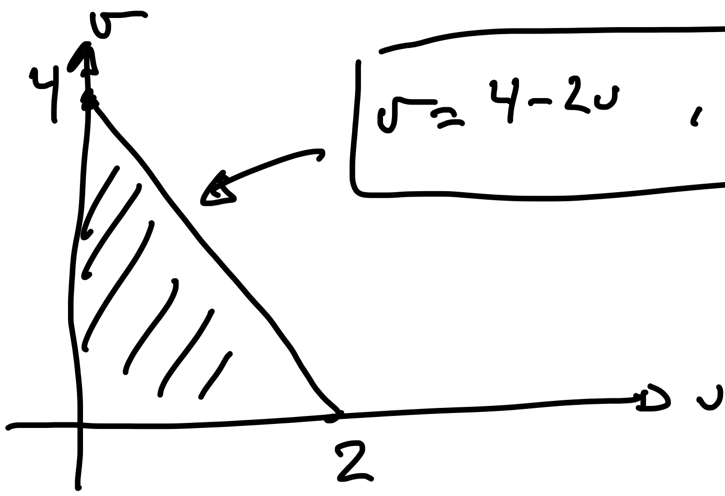
$x = u$

$y = v$

$z = 4 - 2u - v$

Now, our domain D is: $D = \{ (u,v) = 0 \leq u \leq 2, \ 0 \leq v \leq 4-2u \}$

We can sketch this as:



$v = 4-2u$, $\therefore$ $0 \le v \le 4-2u$.

$\mathbf{r}_u(u,v) = \langle 1, 0, -2 \rangle$.

$\mathbf{r}_v(u,v) = \langle 0, 1, -1 \rangle$.

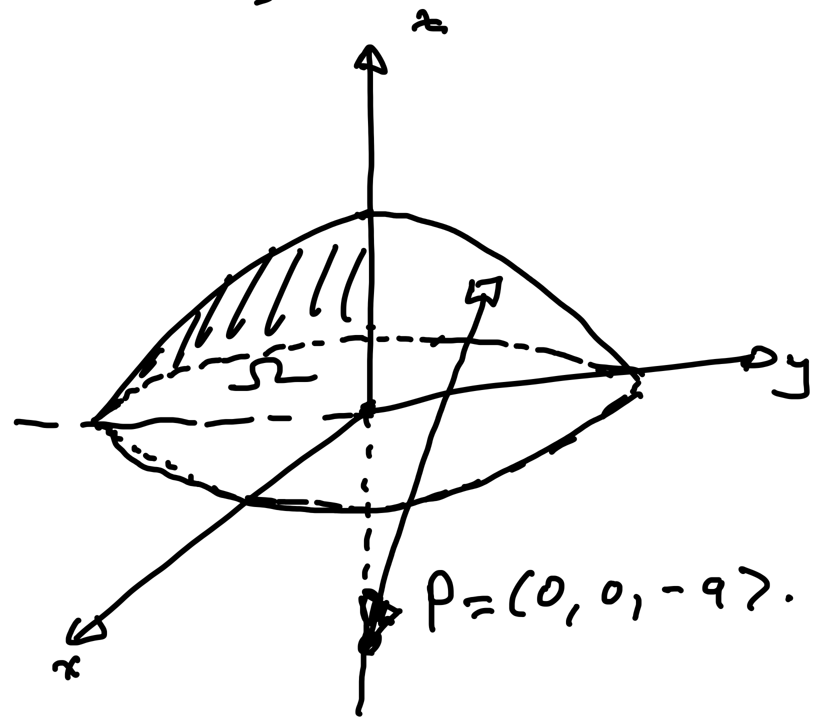
$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$$

$$\therefore \iint_{\Omega} x^2 \, d\Omega = \int_0^2 \int_0^{4-2u} \sqrt{6}\, u^2 \, dv \, du$$

$$= \sqrt{6} \int_0^2 u^2 (4-2u) \, du$$

$$= \sqrt{6} \left[\frac{4u^3}{3} - \frac{u^4}{2} \right]_0^2 = \sqrt{6} \left[\frac{32}{3} - 8 \right]$$

Ex. Find the electric potential at point "p" from a uniformly charged surface $\Omega = \{ x^2 + y^2 + z^2 = a^2,\ z \geq 0 \}$



Electric field potential:

$$V_p = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|} = \frac{1}{4\pi\varepsilon_0} \iint_{\Omega} \frac{\sigma}{|\mathbf{r} - \mathbf{r}_i|} \, d\Omega.$$

q_i = charge density. $\qquad \mathbf{r}_i$ = point

σ = charge density / Area.

① Parametrize the hemisphere:

$$\mathbf{r}(u,v) = \langle a \cos u \sin v,\ a \sin u \sin v,\ a \cos v \rangle.$$

$$0 \leq u \leq 2\pi$$

$$0 \leq v \leq \pi/2$$

Now,

$$|\mathbf{r}-\mathbf{r}_i| = \sqrt{a^2\cos^2 u\sin^2 v + a^2\sin^2 u\sin^2 v + (a\cos v-(-a))^2}$$

W.T.S ! $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$; $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

So

$$|\mathbf{r}-\mathbf{r}_i| = \sqrt{2a^2(1+\cos v)}$$

Now, lets go back to the surface integral:

$$V_p = \frac{1}{4\pi\varepsilon_0}\iint_{\Omega} \frac{\sigma}{\sqrt{2a^2(1+\cos v)}}\, d\Omega$$

$d\Omega$ =

$$V_p = \frac{1}{4\pi\varepsilon_0}\iint_{D} \frac{\sigma\,|\mathbf{r}_u\times\mathbf{r}_v|}{\sqrt{2a^2(1+\cos v)}}\, dA$$

$\mathbf{r}_u(u,v) = \langle -a\sin u\sin v,\ a\cos u\sin v,\ 0\rangle$

$\mathbf{r}_v(u,v) = \langle a\cos u\cos v,\ a\sin u\cos v,\ -a\sin v\rangle.$

Here, $|\mathbf{r}_u\times\mathbf{r}_v| = a^2\sin v$

Now, we can integrate!

$$V_P = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi/2} \frac{\sigma a^2 \sin\theta}{\sqrt{2a^2(1+\cos\theta)}} \, d\theta \, d\phi$$

let $m = 1 + \cos\theta$

$dm = -\sin\theta \, d\theta$

Hence

$$V_P = \frac{2\sigma a}{4\sqrt{2}\,\varepsilon_0} \int_2^1 \frac{-dm}{\sqrt{m}} = \frac{a\sigma}{2\sqrt{2}\,\varepsilon_0} \Big[-2m^{1/2} \Big]_2^1$$

$$= \boxed{\frac{a\sigma}{\sqrt{2}} \Big[\sqrt{2} - 1 \Big]}$$