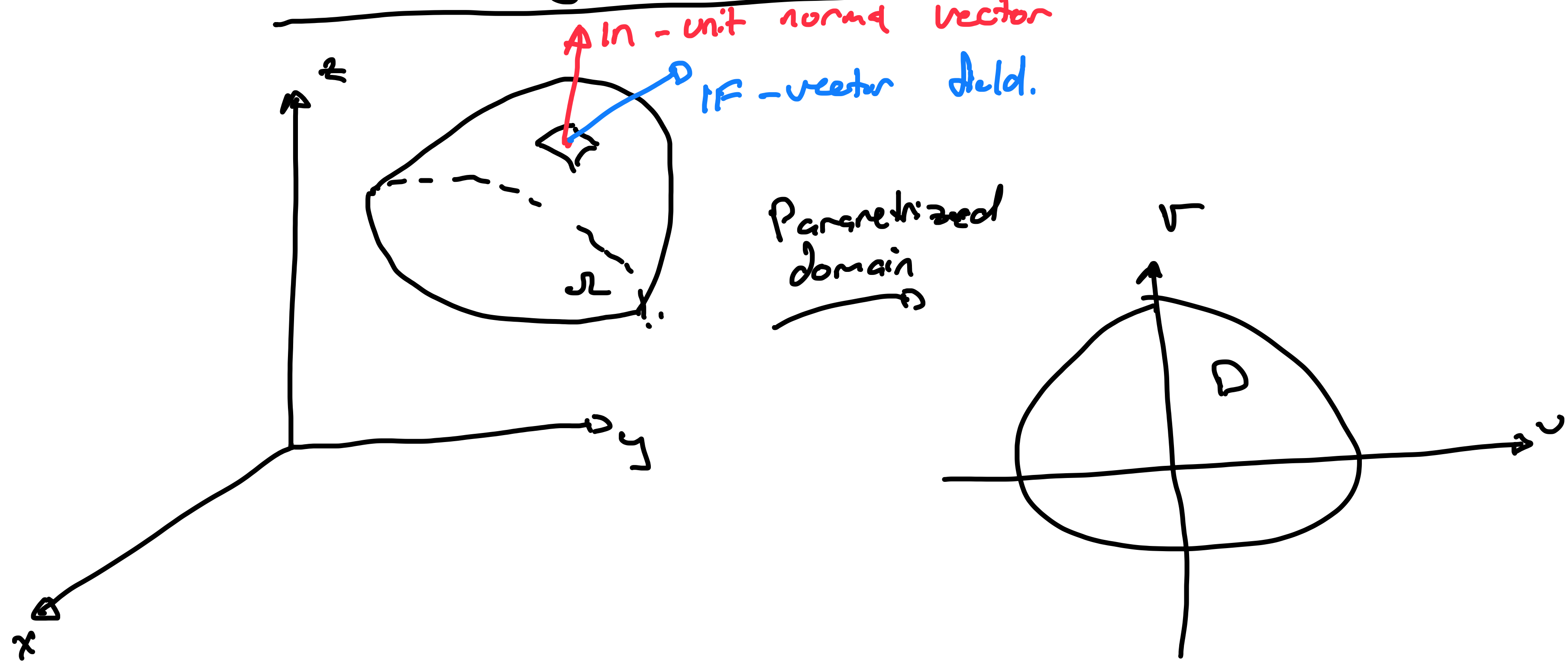
Surface integrals of vector fields



Lets consider a continuous vector field $\mathbb{F}$. To find the flux of $\mathbb{F}$ through a surface Ω, we need the unit normal vector pointing outwards,

$$\boxed{\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}}$$

Here, the flux of $\mathbb{F}$ through a surface element is : $\mathbb{F} \cdot \mathbf{n}\, \Delta\Omega \Longrightarrow \mathbb{F} \cdot \mathbf{n}\, d\Omega$ as $\Delta\Omega \to 0$.

$\therefore$ The flux of $\mathbb{F}$ over all Ω, is simply:

$$\iint_{\Omega} \mathbb{F} \cdot \mathbf{n}\, d\Omega \text{ , where } d\Omega = |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

Subing this into our integral, we get an expression for the flux of $\mathbb{F}$ through Ω in terms of our parametrized variables:

$$\boxed{\iint_{\Omega} \mathbb{F} \cdot \mathbb{n} \, d\Omega = \iint_{D} \mathbb{F}(\mathbb{r}(u,v)) \cdot (\mathbb{r}_u \times \mathbb{r}_v) \, dA}$$

Ex.

$\mathbb{F}(x,y,z) = \langle y, -x, z^2 \rangle$

$\Omega : \mathbb{r}(u,v) = \langle u\cos v, u\sin v, v \rangle.$

AND $D = \{ 0 \leq u \leq 1, \; 0 \leq v, \; \pi/2 \}$

Compute the flux of $\mathbb{F}$ through Ω.

Let $\Phi = \iint_{\Omega} \mathbb{F} \cdot \mathbb{n} \, d\Omega = \iint_{D} \mathbb{F}(\mathbb{r}(u,v)) \cdot (\mathbb{r}_u \times \mathbb{r}_v) \, dA.$

First, find $\mathbb{r}_u$ and $\mathbb{r}_v$.

$\mathbb{r}_u(u,v) = \langle \cos v, \sin v, 0 \rangle$

$\mathbb{r}_v(u,v) = \langle -u\sin v, u\cos v, 1 \rangle.$

Here, $\mathbb{r}_u \times \mathbb{r}_v = \langle \sin v, -\cos v, u \rangle.$

$$\Phi = \iint_D \mathbb{F}(\mathbf{r}(u,v)) \cdot \langle \sin v, -\cos v, u \rangle \, dA.$$

$$= \int_0^{\pi/2}\int_0^1 \langle u\sin v, -u\cos v, v^2 \rangle \cdot \langle \sin v, -\cos v, u \rangle \, du\, dv$$

$$= \int_0^{\pi/2}\int_0^1 u\sin^2 v + u\cos^2 v + uv^2 \, du\, dv$$

$$= \int_0^{\pi/2} \frac{1}{2} + \frac{v^2}{2} \, dv$$

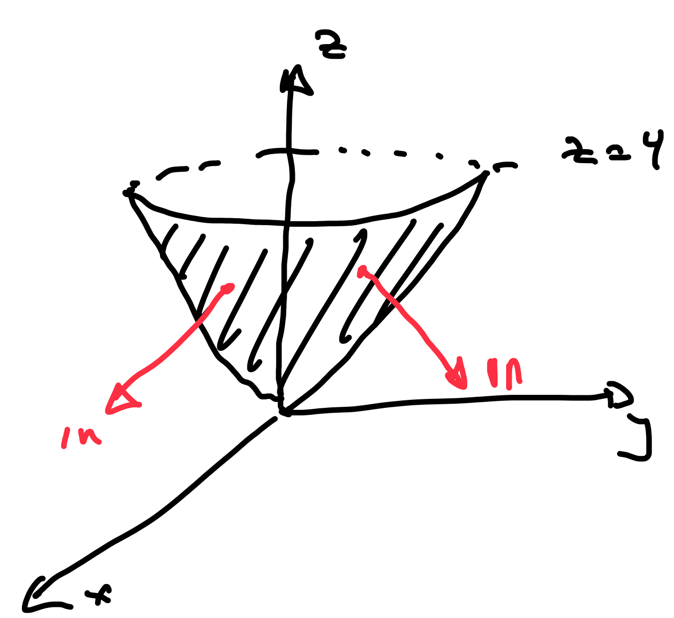
$$= \boxed{\frac{\pi}{4} + \frac{\pi^3}{48}}$$

<u>Ex.</u>

Find the flux integral of:

$\mathbb{F}(x,y,z) = \langle y^3, x^3, 3z^2 \rangle$, through the surface!

$\Omega = \{ x^2+y^2 = z, \quad z \le 4 \}$



Let's parametrize the surface:

$x = \sqrt{v}\cos u$
$y = \sqrt{v}\sin u$
$z = v$

$\mathbf{r}(u,v) = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, v \rangle.$

Also

$\mathbf{r}_u(u,v) = \langle -\sqrt{v}\sin u, \sqrt{v}\cos u, 0 \rangle.$

$\mathbf{r}_v(u,v) = \langle \tfrac{1}{2}(v)^{-1/2}\cos u, \tfrac{1}{2}(v)^{-1/2}\sin u, 1 \rangle.$

$\mathbf{r}_u \times \mathbf{r}_v = \langle \sqrt{v}\cos u, \sqrt{v}\sin u, -\tfrac{1}{2} \rangle.$

[outwards normal vector]

Here

$$\Phi = \iint_D \langle v^{3/2}\sin^3 u, v^{3/2}\cos^3 u, 3v^2 \rangle \cdot \langle \sqrt{v}\cos u, \sqrt{v}\sin u, -\tfrac{1}{2} \rangle \, dA$$

$$= \int_0^{2\pi}\int_0^4 v^2 \sin u\cos u - \frac{3v^2}{2} \, dv\,du$$

$$= \int_0^{2\pi} \frac{64}{3}\sin u\cos u - 32 \, du$$

$$= \boxed{-64\pi}$$