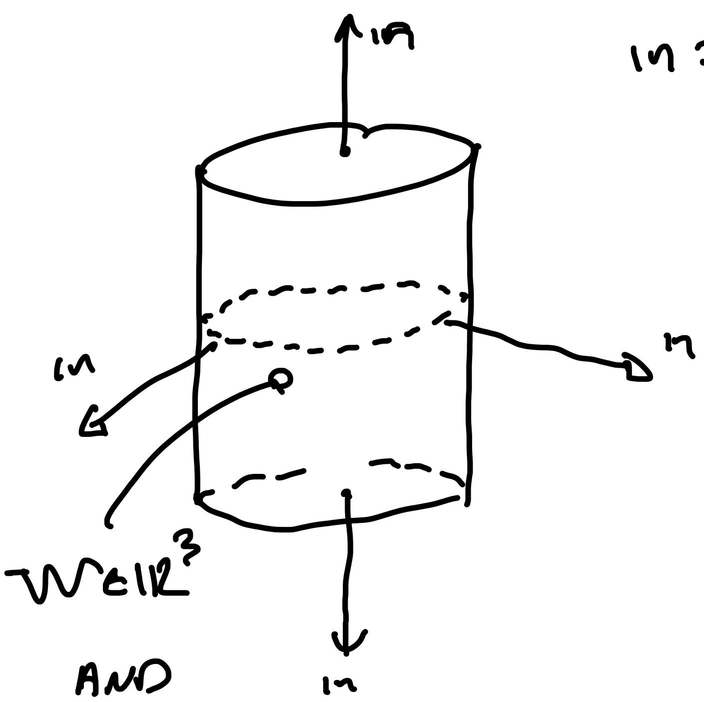
Surface integrals of vector fields and their relation to the Divergence Theorem.



$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

AND

$\Omega = \partial \mathcal{W}$
(i.e. closed surface)

- For a continuously differentiable and smooth vector field $\mathbf{F}$, the flux integral:

$$\underbrace{\iint_{\Omega = \partial \mathcal{W}} \mathbf{F} \cdot \mathbf{n} \, d\Omega}_{\text{Flux integral through the volume.}} = \iiint_{\mathcal{W}} (\nabla \cdot \mathbf{F}) \, dV$$

- We used the Divergence Theorem to find the flux integral through a cylinder in Lecture 13..

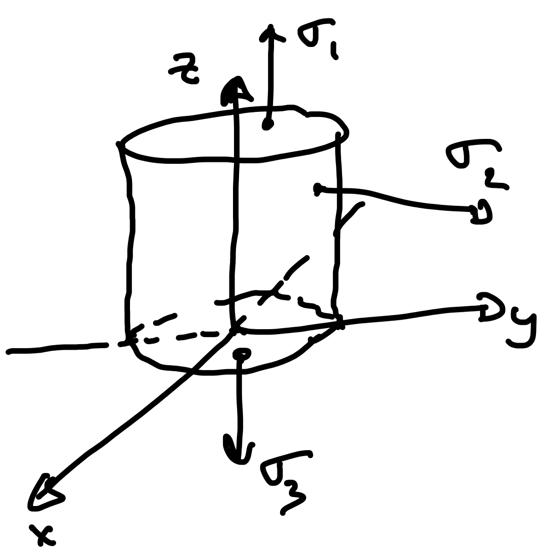
Now, lets solve the same problem but calculate the flux through each surface and see if they match.

The problem was:

Find the flux of fluid through a cylinder with closed ends:
$\Omega = \{ x^2+y^2 = 9 \ , \ 0 \le z \le 5 \}$, where the vector field is described by: $\mathbb{F}(x,y,z) = \langle x^3, y^3, x^3+y^3 \rangle$.

We already know that: $\Phi = \iiint\limits_{W} (\nabla \cdot \mathbb{F})\, dV = \frac{1215\pi}{2}$, so

now let's calculate the flux through each surfaces:



$\sigma_1 := \{ x^2+y^2 \le 9, \ z=5 \}$

$\sigma_2 := \{ x^2+y^2 = 9, \ 0 \le z \le 5 \}$

$\sigma_3 := \{ x^2+y^2 \le 9 \ , \ z=0 \}$

Hence,

$$\iint\limits_{\Omega} \mathbb{F}\cdot\mathbb{n}\, d\Omega = \iint\limits_{\sigma_1} \mathbb{F}\cdot\mathbb{n}\, d\sigma_1 + \iint\limits_{\sigma_2} \mathbb{F}\cdot\mathbb{n}\, d\sigma_2 + \iint\limits_{\sigma_3} \mathbb{F}\cdot\mathbb{n}\, d\sigma_3 .$$

$\underline{\sigma_1}$

$\sigma_1 := \{ x^2+y^2 \le 9, \quad z=5 \}$

$x = v\cos u$
$y = v\sin u$
$z = 5$

$\Rightarrow \quad \mathbf{r}(u,v) = \langle v\cos u, v\sin u, 5 \rangle.$

AND

$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -1 \rangle.$

Here,

$$\Phi(\sigma_1) = \iint_D \langle v^3\cos^3 u, v^3\sin^3 u, v^3(\cos^3 u + \sin^3 u) \rangle \cdot \langle 0,0,-1 \rangle \, dA.$$

$$= \int_0^{2\pi}\int_0^3 -v^3 \underbrace{(\cos^3 u + \sin^3 u)}_{0.} \, dA.$$

$$= 0$$

Note that $\Phi(\sigma_1) = \Phi(\sigma_3) = 0.$

$\underline{\underline{\sigma_2}}$

$\sigma_2 := \{ x^2+y^2 = 9, \quad 0 \le z \le 5 \}$

$x = 3\cos u$
$y = 3\sin u$
$z = v$

$\Rightarrow \quad \mathbf{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle.$

$\mathbf{r}_u \times \mathbf{r}_v = \langle 3\cos u, 3\sin u, 0 \rangle.$

Hence,

$$\Phi(\sigma_3) = \iint_D \langle 27\cos^3 u, 27\sin^3 u, 27(\cos^3 u + \sin^3 u) \rangle \cdot \langle 3\cos u, 3\sin u, 0 \rangle \, dA$$

$$= \iint_D 81\cos^4 u + 81\sin^4 u \, dA.$$

$$= 81 \int_0^{2\pi}\int_0^{5} \cos^4 u + \sin^4 u \, dv\, du$$

$$= 405 \int_0^{2\pi} \cos^4 u + \sin^4 u \, du$$

$$\cos^4 u = \frac{3 + 4\cos 2u + \cos 4u}{8}$$

$$\sin^4 u = \frac{3 - 4\cos 2u + \cos 4u}{8}$$

Hence

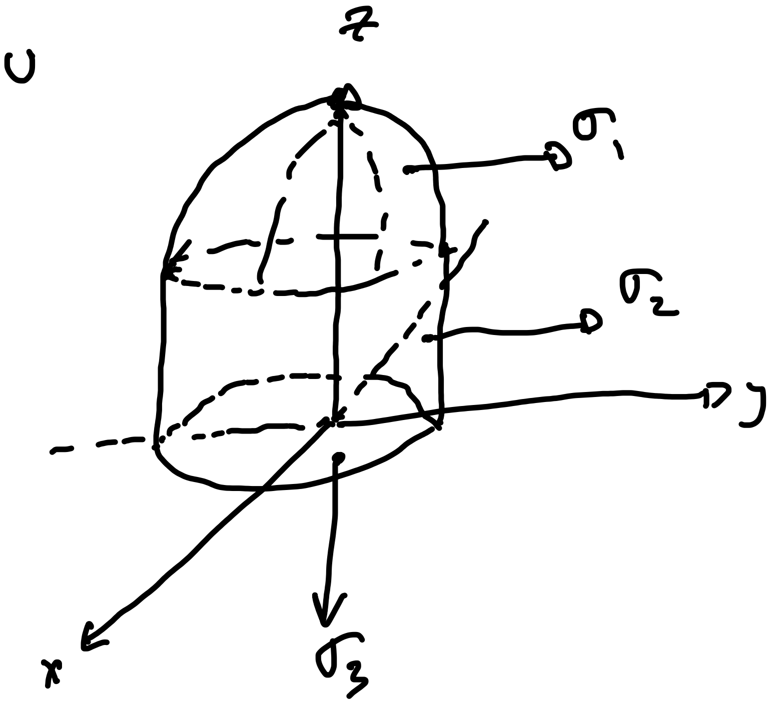
$$\Phi(\sigma_3) = \frac{405}{8}\int_0^{2\pi} 6 + \underbrace{\cancel{2\cos 4u}}_{\to 0} \, du$$

$$= \frac{405}{8}(12\pi) = \boxed{\frac{1215\pi}{2}}$$

Ex.

Find the flux of fluid through the object:

$$\Omega = \{3x^2+3y^2+z=4,\quad 1 \le z \le 4\} \cup$$
$$\{x^2+y^2=1,\quad 0 \le z \le 1\} \cup$$
$$\{x^2+y^2 \le 1,\quad z=0\}$$



the vector field is described by:

$$\mathbb{F}(x,y,z) = \langle xy, -\tfrac{1}{2}y^2, z \rangle.$$

Lets first use the Divergence Theorem:

$$\Phi = \iiint_V \nabla \cdot \mathbb{F}\, dV.$$

$$\nabla \cdot \mathbb{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(-\tfrac{1}{2}y^2\right) + \frac{\partial}{\partial z}(z)$$

$$= 1$$

To integrate, let:

$r^2 = x^2 + y^2$ and

$z = 4 - 3r^2$

Hence,

$$\Phi = \iiint_V r\,\partial z\,\partial r\,\partial\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} r\,\partial z\,\partial r\,\partial\theta$$

$$= 2\pi \int_0^1 r(4-3r^2)\,\partial r = 2\pi\left[2r^2 - \frac{3}{4}r^4\right]_0^1$$

$$= \boxed{\frac{5\pi}{2}}$$

- Now, let's try it by surface integration.

$\underline{\underline{\sigma_1}}$

Parametrize with:

$x = u$

$y = v$

$z = 4 - 3(u^2+v^2)$

$\Rightarrow$

$\mathbf{r}_u = \langle 1, 0, -6u \rangle.$

$\mathbf{r}_v = \langle 0, 1, -6v \rangle.$

$\mathbf{r}_u \times \mathbf{r}_v = \langle 6u, 6v, 1 \rangle$

Hence

$$\Phi(\sigma_1) = \iint_D \langle uv, \tfrac{1}{2}v^2, 4-3(u^2+v^2) \rangle \cdot \langle 6u, 6v, 1 \rangle \, dA.$$

$$= \iint_D 6u^2v + 3v^3 + 4 - 3(u^2+v^2) \, dA.$$

Switch to polar coordinates:

$r^2 = u^2+v^2$

$u = r\cos\theta$

$v = r\sin\theta.$

$$\Phi(\sigma_1) = \int_0^{2\pi}\int_0^1 \underbrace{6r^4\cos^2\theta\sin\theta}_{0} + \underbrace{3r^4\sin^3\theta}_{0} + 4 - 3r^3 \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 4 - 3r^3 \, dr\,d\theta$$

$$= \boxed{5\pi/2}$$

Note that $\Phi(\sigma_2) = \Phi(\sigma_3) = 0$