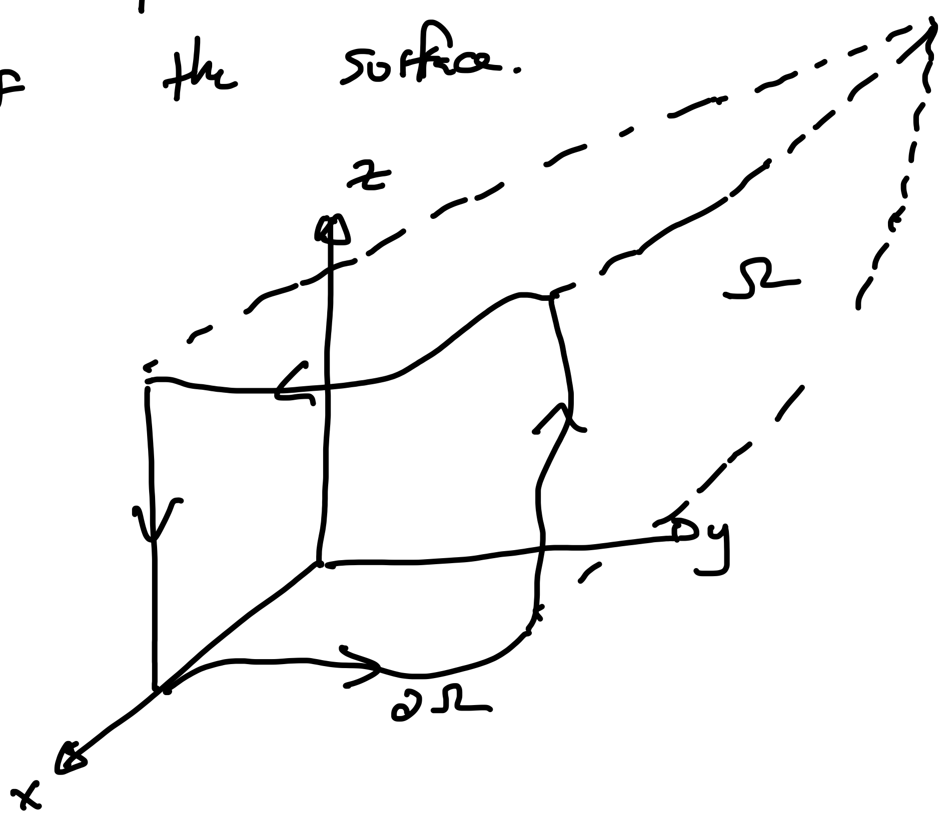
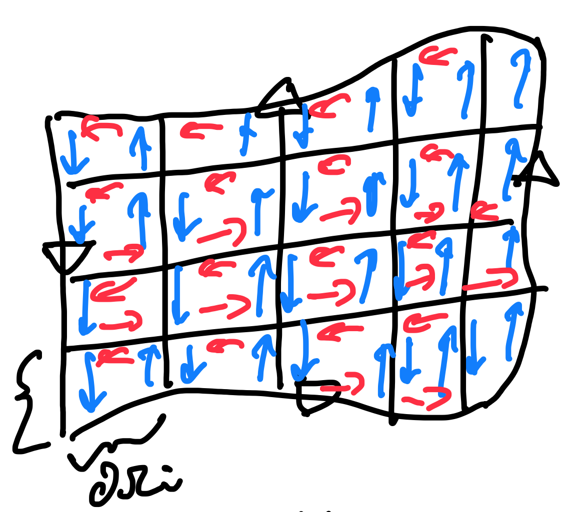
Stoke's Theorem.

- Relates the surface integral of the curl of a vector field with the line integral of that same vector field around the boundary of the surface.



⟶ Let's first focus on the line integral:



All the arrows cancel eachother on shared interface!

Here $\sum_{i=1}^{N} \int \mathbb{F}\cdot d\mathbf{r} = \int_{\partial\Omega} \mathbb{F}\cdot d\mathbf{r}$

$\Rightarrow$ For this C.C.W example (i.e. positively oriented) we can see that, similar to Green's Theorem, the line integral can be approximated with the curl oriented orthogonal to each small piece

$$\int_{\partial\Omega_i} \mathbb{F}\cdot d\mathbb{r} \approx \underbrace{(\nabla\times\mathbb{F})\cdot\mathbb{n}}_{\substack{\text{Component}\\ \text{orthogonal to}\\ \partial\Omega_i}} \underbrace{\partial\Omega_i}_{\text{Area of a small piece.}}$$

Hence, the Stoke's Theorem is just the sum of all the pieces...

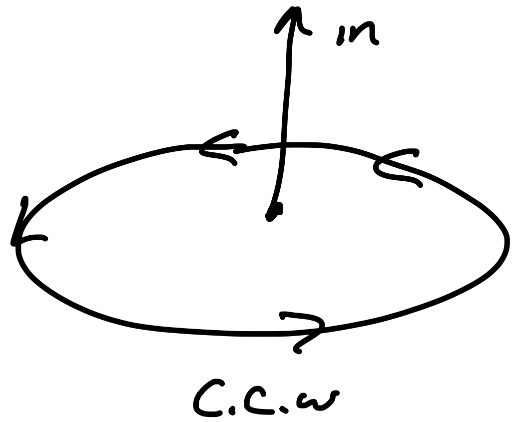
$$\sum_{i=1}^{N} \int_{\partial\Omega_i} \mathbb{F}\cdot d\mathbb{r} = \sum_{i=1}^{N} (\nabla\times\mathbb{F})\cdot\mathbb{n}\,\partial\Omega_i$$

$$\overset{or}{\approx} \text{ as } \partial\Omega_i \to 0$$

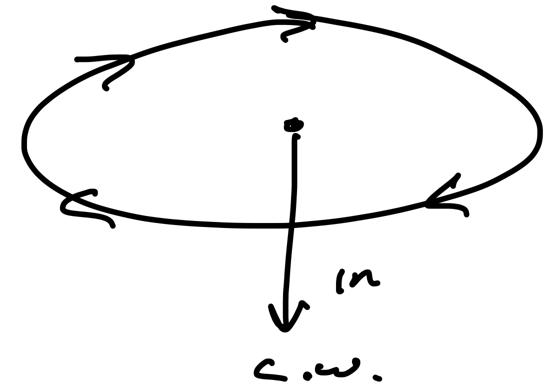
$$\boxed{\int_{\partial\Omega} \mathbb{F}\cdot d\mathbb{r} = \iint_{\Omega} (\nabla\times\mathbb{F})\cdot\mathbb{n}\,\partial\Omega}$$

Note that R.H.R here is very important! Must ensure that your unit normal vector is oriented positively with C.C.W rotation, or negatively with C.W. rotation.

ie.



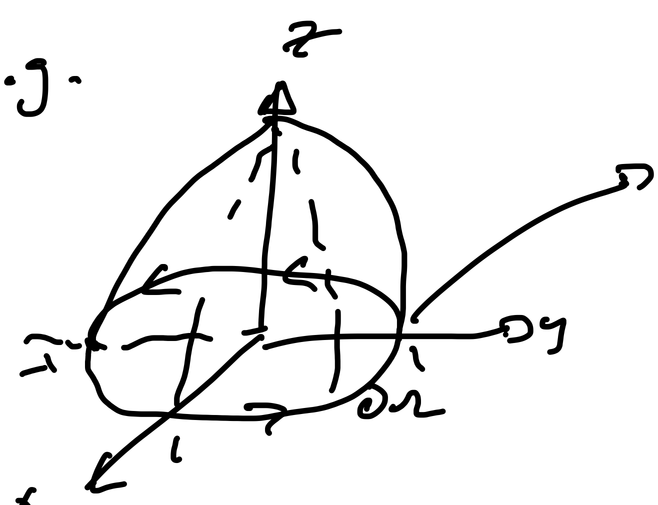
or

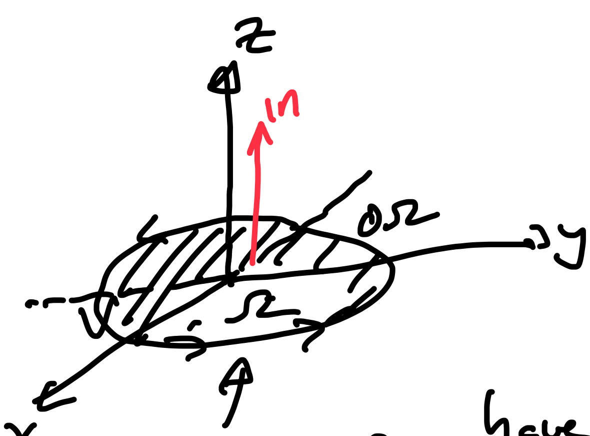


A unique feature.

- Stoke's Theorem is unique in that there are two obvious ways to calculate the surface integral of complex shapes.
- One choice is to project the surface to the plane the 'boundary curve' $\partial\Omega$ creates.

e.g.





Now, we have a surface that's a circle of radius 1.

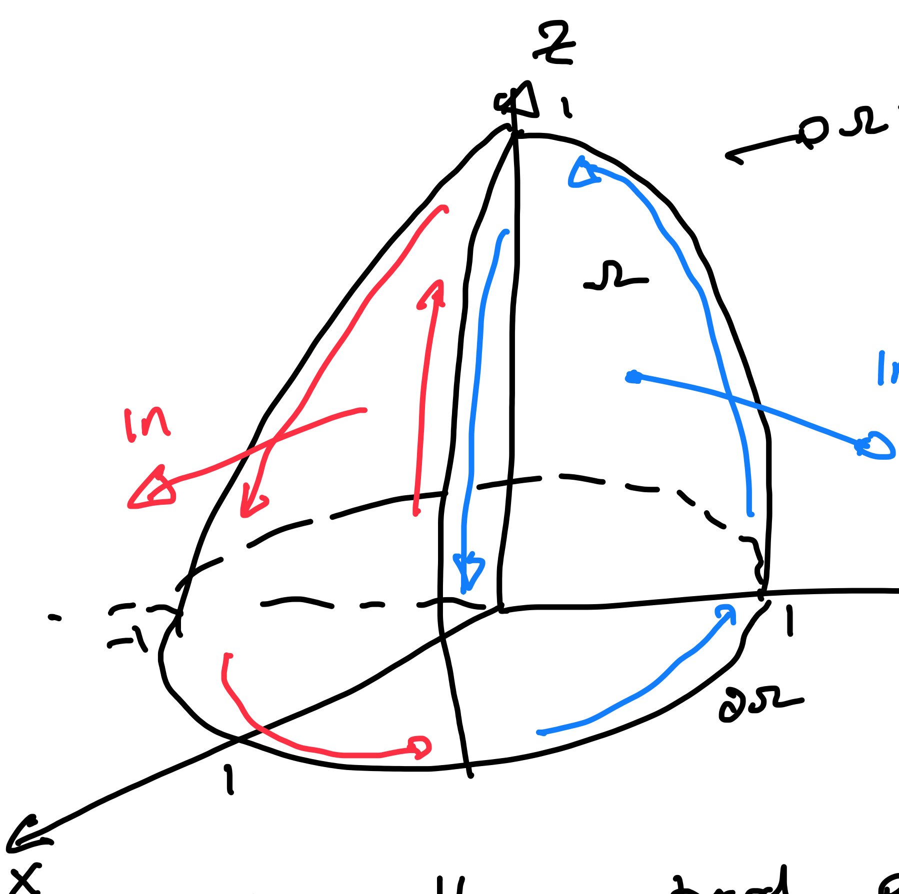
Parametrization $\rightarrow$ $x = r\sin u$
$y = r\cos u$
$z = 0.$

$\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_r}{|\mathbf{r}_u \times \mathbf{r}_r|} = \dfrac{\langle 0,0,r\rangle}{r} = \langle 0,0,1\rangle$ (checks out)

Option #2.

We cut the hemisphere into sections instead of the plane that's bound within the boundary curve.

$$\partial\Omega = \{x^2+y^2+z^2=1 \quad z \geq 0\}$$



Here, the outward pointing normal vector is found by parametrizing the surface in an identical fashion to what we've done before.

$$x = \sqrt{1-v^2}\cos u$$
$$y = \sqrt{1-v^2}\sin u$$
$$z = v$$

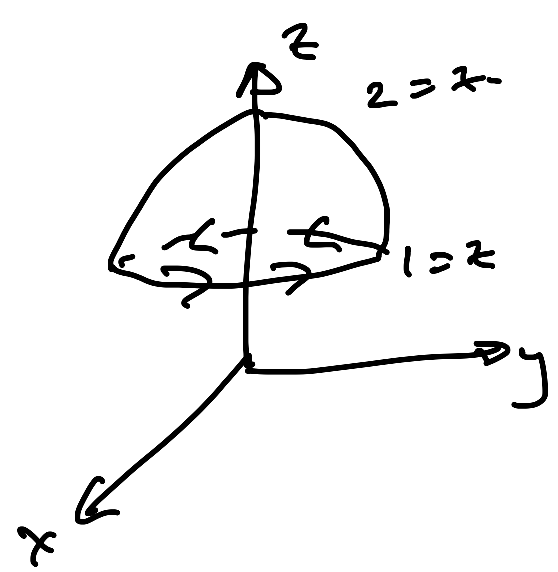
$$\boxed{\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \langle \sqrt{1-v^2}\cos u, \sqrt{1-v^2}\sin u, v \rangle.}$$

These two methods yield the same result for $\iint_{\Omega} (\nabla \times \mathbb{F}) \cdot \mathbf{n}\, \partial\Omega$, since they're just two ways of breaking up the same line integral $\partial\Omega$, ie. the boundary curve.

Let's prove this with an example

Use Stoke's theorem to evaluate:

$\mathbb{F}(x,y,z) = \langle 2, x, y^2 \rangle$ $\quad \Omega = \{ 2 - x^2 - y^2 = z, \quad 1 \le z \le 2 \}$



First let's find the line integral. At $z=1$, we get:

$x = \cos t$
$y = \sin t$
$z = 1$
$\rightarrow \mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$.

Hence,

$$\int_{\partial\Omega} \mathbb{F} \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \langle 2, \cos t, \sin^2 t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt.$$

$$= \int_0^{2\pi} -2\cancelto{0}{\sin t} + \cos^2 t \, dt$$

$$= \int_0^{2\pi} \frac{1 + \cos(2t)}{2} \, dt = \pi$$

Let's parametrize the circle that is contained within the boundary curve $\partial\Omega$.

$x = v \sin u$
$y = v \cos u$
$z = 1$

Ensure's positive orientation!!
R.H.R.

$\mathbf{r}(u,v) = \langle v \sin u, v \cos u, 1 \rangle$.

$\mathbf{r}_u = \langle v \cos u, -v \sin u, 0 \rangle$.

$\mathbf{r}_v = \langle \sin u, \cos u, 0 \rangle$.

$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, v \rangle$. (upwards / positive).

AND

$\nabla \times \mathbf{F} = \langle 2y, 0, 1 \rangle$.

Hence, $\int_0^{2\pi}\int_0^{1} \langle 2v \cos u, 0, 1 \rangle \cdot \langle 0, 0, v \rangle \, du\,dv$.

$= \boxed{\pi}$

Now, the last method...

Parametrize the surface with

$$x = \sqrt{2-v}\cos u, \quad y = \sqrt{2-v}\sin u, \quad z = v$$

$\rightarrow$

$$\mathbf{r}_u = \langle -\sqrt{2-v}\sin u, \sqrt{2-v}\cos u, 0\rangle.$$
$$\mathbf{r}_v = \langle -\tfrac{1}{2}(2-v)^{-1/2}\cos u, -\tfrac{1}{2}(2-v)^{-1/2}\sin u, 1\rangle.$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sqrt{2-v}\cos u, \sqrt{2-v}\sin u, \tfrac{1}{2}\rangle.$$

Hence,

$$\int_0^{2\pi}\int_1^2 \langle 2\sqrt{2-v}\sin u, 0, 1\rangle \cdot \langle \sqrt{2-v}\cos u, \sqrt{2-v}\sin u, \tfrac{1}{2}\rangle \, du\, dv$$

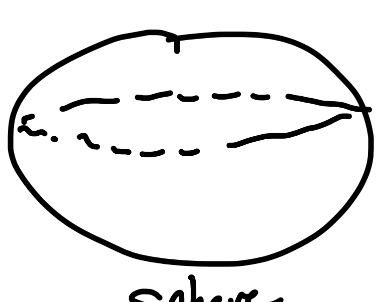
$$= \int_0^{2\pi}\int_1^2 2(2-v)\sin u\cos u + \tfrac{1}{2} \, du\, dv$$

$$= \int_0^{2\pi}\int_1^2 \underbrace{\cancel{(2-v)\sin(2u)}}_{0} + \tfrac{1}{2} \, du\, dv$$

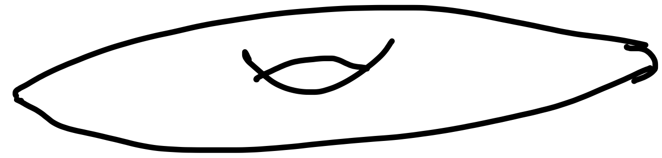
$$= \boxed{\pi}$$

<u>What if there is no boundary curve?</u>

- If you have closed surfaces, e.g.



sphere



torus

All the line integrals cancel and

hence,

$$\boxed{\oint_{\partial\Omega} \mathbb{F}\cdot d\mathbb{r} = 0 = \iint_{\Omega} (\nabla\times\mathbb{F})\cdot \mathbb{n}\, d\Omega}$$