1. Consider $f(x) = \cos(x)$

   By the function, we have $\dfrac{d}{dx}\cos(x) = -\sin(x)$.

   The Taylor series for $\cos(x)$: $\displaystyle\sum_{n\geq 0}\frac{f^{(n)}(0)}{n!}x^n$

   $= f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f^{(3)}(0)}{3!}x^3 + \dfrac{f^{(4)}(0)}{4!}x^4 + \dfrac{f^{(5)}(0)}{5!}x^5 + ...$

   $= \cos(0) - \sin(0)x - \dfrac{\cos(0)}{2!}x^2 + \dfrac{\sin(0)}{3!}x^3 + \dfrac{\cos(0)}{4!}x^4 - \dfrac{\sin(0)}{5!}x^5 + ...$

   $= 1 - 0 - \dfrac{1}{2!}x^2 + 0 + \dfrac{1}{4!}x^4 - 0 + ... = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + ...$

   Taking the derivative,

   $\dfrac{d}{dx}(1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + ...)$

   $= -x + \dfrac{1}{3!}x^3 - \dfrac{1}{5!}x^5 + \dfrac{1}{7!}x^7 - ...$

   $= -(x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + ...)$

   We know that $(x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + ...)$ is the Taylor series for $g(x) = \sin(x)$, because:

   $\displaystyle\sum_{n\geq 0}\frac{g^{(n)}(0)}{n!}x^n$

   $= g(0) + g'(0)x + \dfrac{g''(0)}{2!}x^2 + \dfrac{g^{(3)}(0)}{3!}x^3 + \dfrac{g^{(4)}(0)}{4!}x^4 + \dfrac{g^{(5)}(0)}{5!}x^5 + ...$

   $= \sin(0) + \cos(0)x - \dfrac{\sin(0)}{2!}x^2 - \dfrac{\cos(0)}{3!}x^3 + \dfrac{\sin(0)}{4!}x^4 + \dfrac{\cos(0)}{5!}x^5 + ...$

   $= 0 + x - 0 - \dfrac{1}{3!}x^3 + 0 + \dfrac{1}{5!}x^5 - 0 + ... = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + ...$

   So, by the series, we have $\dfrac{d}{dx}\cos(x) = -(x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + ...) = -\sin(x)$

2. Consider $f(x) = \dfrac{1}{1-x}$

   Get the derivative of this function, we have:

   $\dfrac{d}{dx}f(x) = \dfrac{d}{dx}(\dfrac{1}{1-x}) = \dfrac{1}{(1-x)^2}$

   The Taylor series for $f(x) = \frac{1}{1-x}$ is:

   $\displaystyle\sum_{n\geq 0}\frac{f^{(n)}(0)}{n!}x^n$

   $= f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f^{(3)}(0)}{3!}x^3 + \dfrac{f^{(4)}(0)}{4!}x^4 + \dfrac{f^{(5)}(0)}{5!}x^5 + ...$

   $= 1 + x + x^2 + x^3 + x^4 + ...$

   $= \displaystyle\sum_{n\geq 0} x^n$

   Take the derivative of $\displaystyle\sum_{n\geq 0} x^n$, then we can get :

   $\dfrac{d}{dx}(\displaystyle\sum_{n\geq 0} x^n) = \dfrac{d}{dx}(1 + x + x^2 + x^3 + x^4 + ...) = 0 + 1 + 2x + 3x^2 + 4x^3 + ... = \displaystyle\sum_{n\geq 0} nx^{n-1}$

We know that $\sum_{n\geq 0} nx^{n-1}$ is the Taylor series for $g(x) = \frac{1}{(1-x)^2}$,because:

$$\sum_{n\geq 0} \frac{g^{(n)}(0)}{n!}x^n$$
$$= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g^{(3)}(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 + \frac{g^{(5)}(0)}{5!}x^5 + ...$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...$$
$$= \sum_{n\geq 0} nx^{n-1}$$

So we have proved that $\frac{d}{dx}(x^n) = \sum_{n\geq 0} nx^{n-1} = \frac{1}{(1-x)^2}$

3. Consider $f(x) = x^3 + 2x + 1$

   Get the derivative of the function, we have:

   $\frac{d}{dx}(x^3 + 2x + 1) = 3x^2 + 2$

   The Taylor series for $f(x) = x^3 + 2x + 1$ is :

   $$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n$$
   $$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + ...$$
   $$= 1 + \frac{2}{1!}x + \frac{0}{2!}x^2 + \frac{6}{3!}x^3 + \frac{0}{4!}x^4 + \frac{0}{5!}x^5 + ...$$
   $$= 1 + 2x + x^3$$

   Take the derivative of $1 + 2x + x^3$, then we can get:

   $\frac{d}{dx}(1 + 2x + x^3) = 2 + 3x^2$

   Therefore $\frac{d}{dx}(1 + 2x + x^3) = 3x^2 + 2$

4. Consider $f(x) = log(1 + x)$

   By the function, we have $\frac{d}{dx}log(1 + x) = \frac{1}{1+x}$

   The Taylor series for $log(1 + x) = \sum_{n\geq 0} \frac{(-1)^{n+1}f^{(n)}(0)}{n}x^n$

   $$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + ...$$
   $$= log(1 + 0) + (\tfrac{1}{1+0})x - \frac{(\frac{1}{(1+0)^2})x^2}{2!} + \frac{(\frac{2}{(1+0)^3})x^3}{3!} - \frac{(\frac{6}{(1+0)^4})x^4}{4!} + \frac{(\frac{24}{(1+0)^5})x^5}{5!} - ....$$
   $$= 0 + x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \tfrac{5}{5}$$

   Taking the derivative, $\frac{d}{dx}(0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{5}{5})$
   $= 1 - x + x^2 - x^3 + x^4 - x^5 + ...$

   We know that $= (1 - x + x^2 - x^3 + x^4 - x^5 + ...)$ is the Taylor series for $g(x) = \frac{1}{1+x}$, because:

   $$\sum_{n\geq 0} \frac{(-1)^{n+1}g^{(n)}(0)}{n}x^n$$

$$= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g^{(3)}(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 + \frac{g^{(5)}(0)}{5!}x^5 + ...$$
$$= \frac{1}{1+0} - \frac{(\frac{1}{(1+0)^2})x}{1!} + \frac{(\frac{2}{(1+0)^3})x^2}{2!} - \frac{(\frac{6}{(1+0)^4})x^3}{3!} + \frac{(\frac{24}{(1+0)^5})x^4}{4!} - \frac{(\frac{120}{(1+0)^6})x^5}{5!} ....$$
$$= 1 - x + x^2 - x^3 + x^4 - x^5 + ...$$

Therefore, $\frac{d}{dx} log(1+x) = 1 - x + x^2 - x^3 + x^4 - x^5 + ... = \frac{1}{1+x}$

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