* Curve Sketching

Strategy for Curve Sketching

1) Determine the domain
2) Determine asymptotes (horizontal / vertical)
3) Intervals of increasing / decreasing ($f'(x) > 0$, $f'(x) < 0$)
4) Local max. / Local min. ($f'(x) = 0$, 2nd derivative test)
5) Concavity / Points of inflection ($f''(x) > 0$, $f''(x) < 0$, $f''(x) = 0$)
6) Tabulate the information acquired from steps 2-5
7) Sketch the curve

Ex. $y = \dfrac{2x^2}{x^2-1}$

step 1) $x \neq \pm 1$, thus domain $= (-\infty, -1), (-1, 1), (1, \infty)$

step 2) $\displaystyle\lim_{x \to \pm\infty} \frac{2x^2}{x^2-1} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = \frac{2}{1} = 2$ i) vertical: $x = \pm 1$ ii) horizontal: $y = 2$

step 3) $f' = 4x(x^2-1)^{-1} - 4x^3(x^2-1)^{-2} = 4x(x^2-1)^{-2}[x^2-1-x^2]$

$= \dfrac{-4x}{(x^2-1)^2}$ i) increasing: $x < 0$ ii) decreasing: $x > 0$

step 4) $f' = 0$ when $x = 0$

using 1st derivative test, $x = 0$ is a local max. $x = 0, y = 0$

step 5) $f'' = -4(x^2-1)^{-2} + 16x^3(x^2-1)^{-3} = 4(x^2-1)^{-3}[4x^2 - (x^2-1)]$

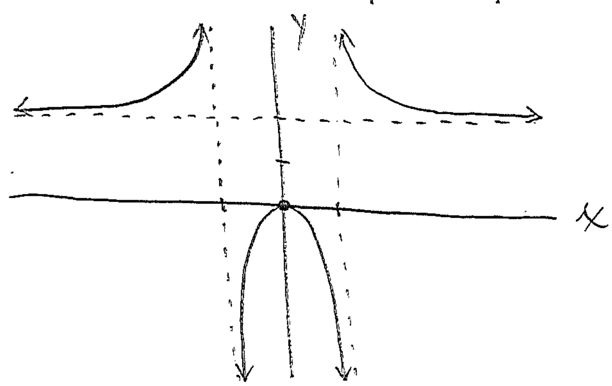
$= \dfrac{12x^2+4}{(x^2-1)^3}$ i) $-1 < x < 1$: $f'' < 0$: CD ii) $x < -1$, $x > 1$: $f'' > 0$: CU

step 6)

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
f'	(+)	(+)	(−)	(−)
f''	CU	CD	CD	CU

vertical asymp. $x = \pm 1$
horizontal asymp. $y = 2$
local max: $(0, 0)$

step 7)



Ch.4.5 Q 1-52
Except #31, 33-3
36-39, 52

Curve Sketching Example 2 (from Dec 2014 Final Exam)

$$f(x) = \frac{e^x}{x^2}$$

Step 1) Domain:

$x \neq 0$, thus domain is $(-\infty, 0), (0, \infty)$

Step 2) Asymptotes:

$\lim_{x \to 0^{\pm}} = +\infty$ $\quad \lim_{x \to \infty} = \infty$ $\quad \lim_{x \to -\infty} = 0$

i) vertical asymptote: $x = 0$

ii) Horizontal asymptote: $\lim_{x \to -\infty} = 0$

Step 3) Intervals of Inc/Dec:

$$f' = \frac{e^x}{x^2} - \frac{2e^x}{x^3} = \frac{e^x}{x^3}(x-2)$$

i) Increasing: $(-\infty, 0), (2, \infty)$

ii) Decreasing: $(0, 2)$

Step 4) Critical Point:

$f' = 0$ at $x = 2$

Using 1st derivative test $(2, \frac{1}{4}e^2)$ is a local minimum.

Step 5) Concavity:

$$f'' = \frac{e^x}{x^2} - \frac{2e^x}{x^3} - \frac{2e^x}{x^3} + \frac{6e^x}{x^4} = \frac{e^x}{x^2} - \frac{4e^x}{x^3} + \frac{6e^x}{x^4}$$

$$f'' = \frac{e^x}{x^4}(x^2 - 4x + 6)$$

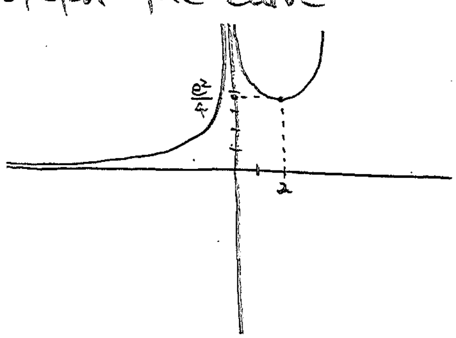
f'' is always positive: CU for $(-\infty, 0), (0, \infty)$

No inflection point

Step 6) Summerize/Tabulate results:

x	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f'	$(+)$	$(-)$	$(+)$
f''	$(+)$	$(+)$	$(+)$

Step 7) Sketch the curve



Curve Sketching Example 3

$$y = \frac{x}{\sqrt{x^2-1}}$$

Step 1) Domain

$(-\infty,-1),\ (1,\infty) \qquad x \neq \pm 1$

Step 2) Asymptotes:

$$\lim_{x\to -1^-} \frac{x}{\sqrt{x^2-1}} = -\infty \qquad \lim_{x\to 1^+} \frac{x}{\sqrt{x^2-1}} = +\infty$$

$$\lim_{x\to -\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{-x}{x\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{-1}{\sqrt{1-1/x^2}} = -1$$

$$\lim_{x\to +\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to+\infty} \frac{x}{x\sqrt{1-1/x^2}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1-1/x^2}} = 1$$

i) Horizontal Asymptote: $x = \pm 1$

ii) Vertical Asymptote: $y = \pm 1$

Step 3) Interval of Inc/Dec

$$y' = \frac{1}{\sqrt{x^2-1}} - \frac{x^2}{(x^2-1)^{3/2}} = \frac{1}{(x^2-1)^{3/2}}\left[x^2-1-x^2\right] = \frac{-1}{(x^2-1)^{3/2}}$$

Increasing: None

Decreasing: $(-\infty,-1),(1,\infty)$

Step 4) Local max/min

$y' \neq 0$ for all x

$\therefore$ No local max/min

Step 5) Concavity

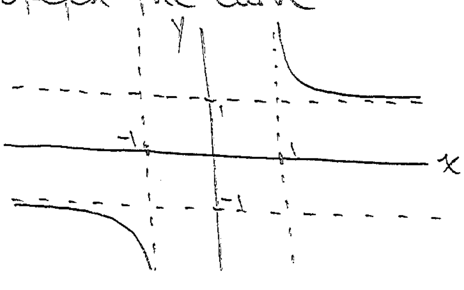
$$y'' = \frac{3}{\cancel{2}}(x^2-1)^{-\frac{5}{2}}\cdot \cancel{2}x = \frac{3x}{(x^2-1)^{5/2}}$$

CU: $(1,\infty)$ $\qquad$ CD: $(-\infty,-1)$

Step 6) Summarize / Tabulate Results:

x	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
f'	$(-)$	X	$(-)$
f''	CD	X	CU

Step 7) Sketch the Curve



Curve Sketching Example 4

$f(x) = e^{-x}\sin x \qquad 0 \le x \le 2\pi$

step 1) Domain:

$f(x) = \dfrac{\sin x}{e^x}$ Denominator + Numerator Defined for all values of x in the interval $[0, 2\pi]$

step 2) Asymptotes

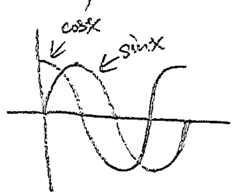
$\lim\limits_{x\to\infty} f(x) = 0 \qquad \lim\limits_{x\to-\infty} \dfrac{\sin x}{e^x} = \lim\limits_{x\to\infty} \dfrac{-\sin x}{e^{-x}} = \lim\limits_{x\to\infty} -e^x \sin x = -\infty$

i) Horizontal asymptotes: $x\to\infty,\ y\to 0$
$x\to-\infty,\ y\to-\infty$

ii) Vertical asymptotes: None

step 3) Intervals of Inc/Dec:

$f'(x) = \cos x \cdot e^{-x} - \sin x \cdot e^{-x} = \dfrac{1}{e^x}(\cos x - \sin x)$



$\cos x - \sin x = 0$
$\sin x = \cos x$
$\tan x = 1$

$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$\therefore$ Inc: $[0, \frac{\pi}{4}), (\frac{5\pi}{4}, 2\pi]$

Dec: $(\frac{\pi}{4}, \frac{5\pi}{4})$

step 4) Critical points:

$f'(x) = \dfrac{\cos x - \sin x}{e^x} = 0 \qquad x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

Using 1st derivative test, $x = \frac{\pi}{4}$ is local max $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2e^{\pi/4}}\right)$

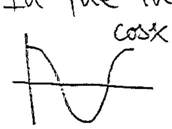
$x = \frac{5\pi}{4}$ is local min $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2e^{5\pi/4}}\right)$

step 5) Concavity:

$f''(x) = \dfrac{-1}{e^x}(\cos x - \sin x) + \dfrac{1}{e^x}(-\sin x - \cos x)$

$= \dfrac{1}{e^x}(-\sin x - \cos x - \cos x + \sin x) = \dfrac{-2}{e^x}(\cos x)$

In the interval $[0, 2\pi]$, $f''(x) = 0$ at $x = \frac{\pi}{2}, \frac{3\pi}{2}$



CU: $(\frac{\pi}{2}, \frac{3\pi}{2})$

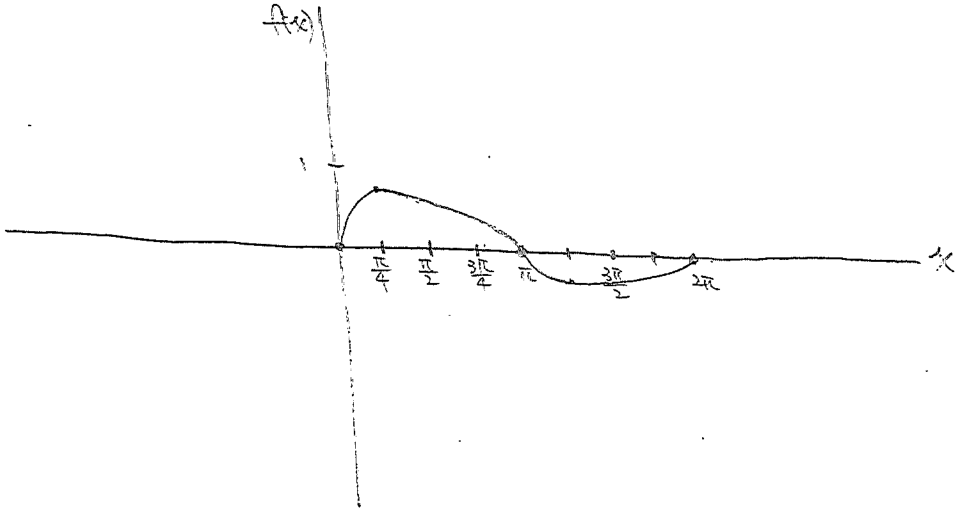
CD: $(0, \frac{\pi}{2}), (\frac{3\pi}{2}, 2\pi)$

Inflection point at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

step 6) Summarize / Tabulate Results

x	$(0, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{5\pi}{4})$	$(\frac{5\pi}{4}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
f'	(+)	(−)	(−)	(+)	(+)
f''	CD	CD	CU	CU	CD

step 7) Sketch the curve



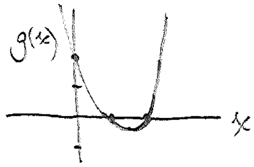
Curve Sketching Example 5

$f(x) = \ln(x^2 - 3x + 2)$

Step 1) Domain:

$\ln(g(x))$ is defined only for $g(x) > 0$

In our case, $g(x) = x^2 - 3x + 2 = (x-1)(x-2)$



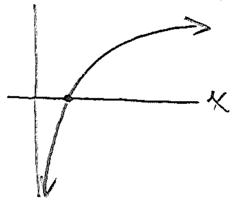
$$g(x) \begin{cases} > 0 & \text{for } (-\infty, 1), (2, \infty) \\ < 0 & \text{for } (1, 2) \end{cases}$$

Since $\ln(g(x))$ is defined only for $g(x) > 0$, our domain is $(-\infty, 1), (2, \infty)$

Step 2) Asymptotes:

$\ln(g(x)) \to -\infty$ when $g(x) \to 0$

Ex. $\ln x$



So, our vertical asymptotes are

$\lim_{x \to 1^-} \ln(x^2 - 3x + 2) = -\infty$ $\quad\quad$ $\lim_{x \to 2^+} \ln(x^2 - 3x + 2) = -\infty$

Step 3) Intervals of Inc/Dec:

$f'(x) = \dfrac{2x-3}{x^2 - 3x + 2} = \dfrac{2x-3}{(x-2)(x-1)}$ $\quad\quad$ $x \neq 1, 2$

$$x = \begin{cases} (+) & x > \frac{3}{2} & (\frac{3}{2}, 2), (2, \infty) & \text{Inc} \\ (-) & x < \frac{3}{2} & (-\infty, 1), (1, \frac{3}{2}) & \text{Dec} \end{cases}$$

However!!! Our Domain tells us that $f(x)$ is defined only in $(-\infty, 1), (2, \infty)$.

Thus,

$$x = \begin{cases} \text{Increasing} & (2, \infty) \\ \text{Decreasing} & (-\infty, 1) \end{cases}$$

Step 4) Local max/min

$f'(x) = \dfrac{2x-3}{(x-2)(x-1)} = 0$ when $x = \frac{3}{2}$

However!!! $x = \frac{3}{2}$ is not within our domain!!!

Thus there is no local max/min

Step 5) Concavity

$$f''(x) = 2(x^2-3x+2)^{-1} - (2x-3)(x^2-3x+2)^{-2}(2x-3)$$
$$= (x^2-3x+2)^{-2}\left[2(x^2-3x+2) - (2x-3)^2\right]$$
$$= \frac{2x^2-6x+4-4x^2+12x-9}{(x^2-3x+2)^2} = \frac{-2x^2+6x-5}{(x^2-3x+2)^2} = \frac{-(2x^2-6x+5)}{(x^2-3x+2)^2}$$

To find concavity:

$$f''(x) = \frac{-(2x^2-6x+5)}{(x^2-3x+2)^2} = 0 \qquad 2x^2-6x+5=0$$

$$x = \frac{6 \pm \sqrt{6^2-4\cdot2\cdot5}}{4} = \frac{6\pm\sqrt{36-40}}{4} = \frac{6\pm\sqrt{-4}}{4}$$ No solution!

$$f''(x) = \frac{-(2x^2-6x+5)}{(x^2-3x+2)^2}$$ Numerator always (-), Denominator always (+) } $f''(x)$ is always (-)

Thus, $f(x) = \ln(x^2-3x+2)$ is always concave down

Step 6) Tabulate info:

x	$(-\infty, 1)$	$(1,2)$	$(2,\infty)$
$f'(x)$	Dec	X	Inc
$f''(x)$	CD	X	CD

$-\infty$ $-\infty$

Step 7) Sketch graph