Review:

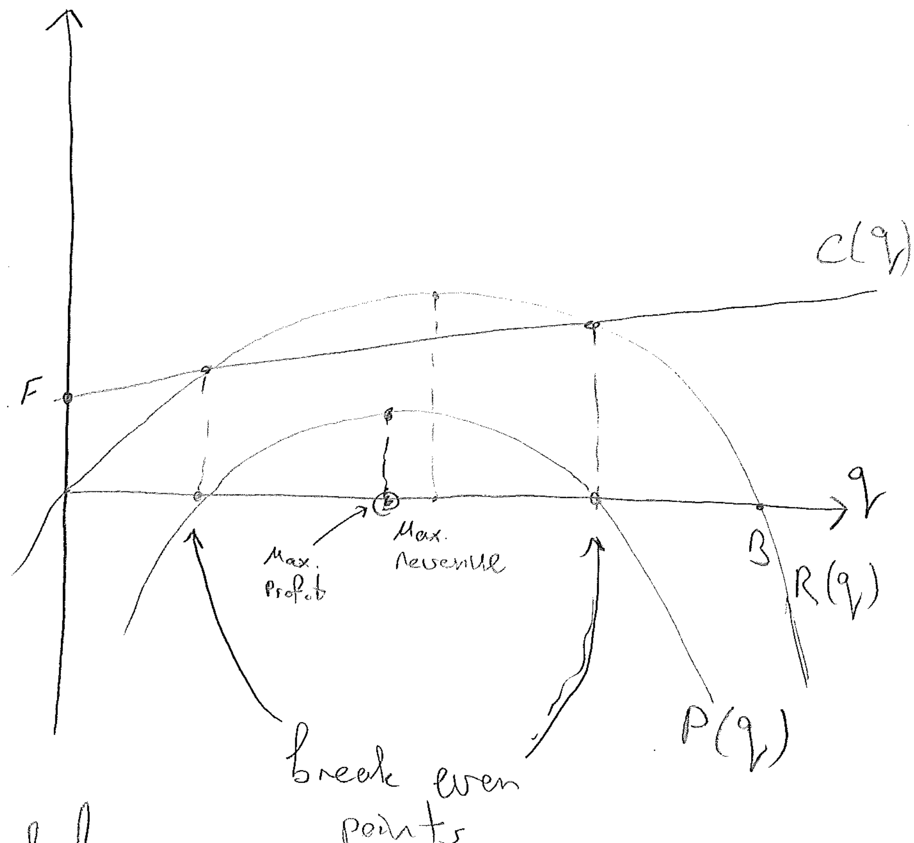
Last time we discussed: revenue, cost, profit, demand.

$$\begin{cases} R = p \cdot q \\ C(q) = F + V(q) \\ P = R - C \end{cases}$$

Linear model: $q = Ap + B$

$$p = \frac{q - B}{A}$$

$(V(q) = D \cdot q)$



If the cost model and demand model were not linear it would have been much harder to find the max. point.

Compound Interest

This example will give more motivation for the use of calculus in economics.

Consider an investment of \$100 for one year at 10%. What ~~is~~ will be the future balance (one year from now)?

ⓐ \$110 ⓑ \$110.25 ⓒ \$110.38 ⓓ \$110.47 ⓔ \$110.516 ⓕ \$110.517
ⓖ it depends ⓗ I don't know

Answer: All are true
It depends!

calculator ready answer

ⓐ If the interest is compounded annually:

$$\underbrace{FV}_{\text{future value}} = \underbrace{\overset{\$100}{PV}}_{\text{past value}} (1+0.1) = \$110$$

ⓑ If the interest is compounded semi-annually:

$$FV = PV \cdot \left(1+\frac{0.1}{2}\right)^2 = \$110.25$$

ⓒ If the interest is compounded quarterly:

$$FV = PV \cdot \left(1+\frac{0.1}{4}\right)^4 = \$110.38$$

ⓓ If the interest is compounded monthly:

$$FV = PV \cdot \left(1+\frac{0.1}{12}\right)^{12} \approx \$110.47$$

ⓔ If the interest is compounded daily:

$$FV = PV \cdot \left(1+\frac{0.1}{365}\right)^{365} \approx \$110.516$$

(f) If the interest is compounded continuously: (divide days to hours, hours to minutes, minutes to seconds and so on...)

First note that: $FV_n = PV \cdot (1 + \frac{0.1}{n})^n$ increase with n.

Can it grow indefenetly? Can we scheme to get any amount of money we want by increasing n?

Well, no.
$$FV = \lim_{n \to \infty} \$100 \left(1 + \frac{0.1}{n}\right)^n = \$100 \cdot e^{0.1} \approx \$110.517$$

This was discovered by Jacob Bernoulli (in 1683).

Definition: $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$

Don't mind about what this means precisely you get the point. (We'll talk about how to compute it later on)

Key terms and Formulas: (Wikipedia: compounded interest)

j = effective interest rate.
i = nominal interest rate
r = real interest rate.

t = time

- Simple interest rate: $FV = PV(1 + j \cdot t)$.
  You don't earn interest on interest
- Effective interest rate: $FV = PV(1+j)^t$.
- Nominal interest rate: $FV = PV\left(1 + \frac{i}{n}\right)^{n \cdot t}$ (for n given).
- Continuous compounding: $FV = PVe^{rt}$.

Equivalent rates: The functions are the same
i.e. equality for any t.

(doesn't work with simple interest)

Effective $\leftrightarrow$ Nominal $\leftrightarrow$ Continuous

Remark: In Canada it is considered criminal to use an effective interest of 60% or more.

Log: $y = e^x \xleftrightarrow{y>0} x = \log(y) = \mathrm{Log}(y) = \ln(y)$

Some rules: (covered in the workshop!):

$e^0 = 1$

$e^{x+y} = e^x \cdot e^y$

$(e^x)^y = e^{x \cdot y}$

$\log(1) = 0$

$\log(x \cdot y) = \log(x) + \log(y)$

$\log(x^y) = y \cdot \log(x)$

$a > 0 \quad a^x = \overset{(a^{\log(e)})^x}{e^{x \cdot \log(a)}}$

$\log_a(y) = \frac{\log(y)}{\log(a)}$

Exercises:

ⓐ You burrow \$50,000 from Nick the shark, who charges you at a fixed rate r that is compounded continuously. If you pay Nick \$100,000 two years later, what was the annual rate of interest that he charged? What was the effective interest rate?

Solution:

$$FV = PV \cdot e^{rt}$$

calculator ready form

$$100{,}000 = 50{,}000 \cdot e^{2r} \qquad (t=2,\ FV=100{,}000,\ PV=50{,}000)$$

$$2 = e^{2r}$$

$$\ln 2 = 2r$$

$$\boxed{r = \frac{\ln 2}{2}} \approx 0{,}347\ldots = 34.7\%$$

$$FV = PV(1+j)^t \rightsquigarrow 100{,}000 = 50{,}000(1+j)^2$$

$$2 = (1+j)^2\ ;\ 1+j = \sqrt{2}\ ;\ \boxed{j = \sqrt{2} - 1} \approx 0.4\ldots$$

ⓑ How many years will it take for \$10,000 to grow to \$12,000 if it is invested at 12% annual interest compounded quarterly?

$$FV = PV\left(1 + \frac{i}{n}\right)^{n \cdot t}$$

$$12{,}000 = 10{,}000\left(1 + \frac{0.12}{4}\right)^{4t}$$

$$1.2 = \left(1 + \frac{0.12}{4}\right)^{4t}$$

$$\ln(1.2) = \ln\left[\left(1 + \frac{0.12}{4}\right)^{4t}\right] = 4t \ln\left(1 + \frac{0.12}{4}\right)$$

$$\boxed{t = \frac{\ln(1.2)}{4\ln\left(1 + \frac{0.12}{4}\right)}}$$

ⓒ What continuously compounded rate is equivalent to 8% compounded semi-monthly.

Solution:

$$\cancel{PV}\cdot e^{rt} = FV = \cancel{PV}\left(1+\frac{0.08}{2}\right)^{2t}$$

$$e^{rt} = (1+0.04)^{2t}$$

$$r\cdot \cancel{t} = \ln\left[(1+0.04)^{2t}\right] = 2\cancel{t}\,\ln(1+0.04)$$

$$\boxed{r = 2\ln(1.04)}$$

ⓓ (A⁺ problem)

A student has a trust currently valued at \$500,000 gets an annual interest rate of 6% compounded continuously. Their plan is to retire once they can withdraw \$10,000 at the beginning of each month indefinitely. How long do they need to wait to retire assuming they don't invest any more money?

Solution:

Step 1: How much money (PV) they need at the time of retirement:

$\underbrace{FV}_{\text{in a month}} = PV + 10{,}000$; $\quad FV = PV\, e^{0.06\cdot\frac{1}{12}}$

$$PV + 10{,}000 = PV\cdot e^{0.06\cdot\frac{1}{12}}$$

$$PV\left(e^{0.06\cdot\frac{1}{12}}-1\right) = 10{,}000$$

$$\boxed{PV = \frac{10{,}000}{e^{0.005}-1}} \approx 1{,}995{,}004.11$$

Step 2:

$$500{,}000\, e^{0.06t} = \frac{10{,}000}{e^{0.005}-1}$$

$$e^{0.06t} = \frac{1}{50\left(e^{0.005}-1\right)}$$

$\frac{6}{100}$

$$0.06t = \ln\left[\frac{1}{50\left(e^{0.005}-1\right)}\right]$$

$$\boxed{t = \frac{100}{6}\ln\left[\frac{1}{50\left(e^{0.005}-1\right)}\right]}$$

≈ 23 years.