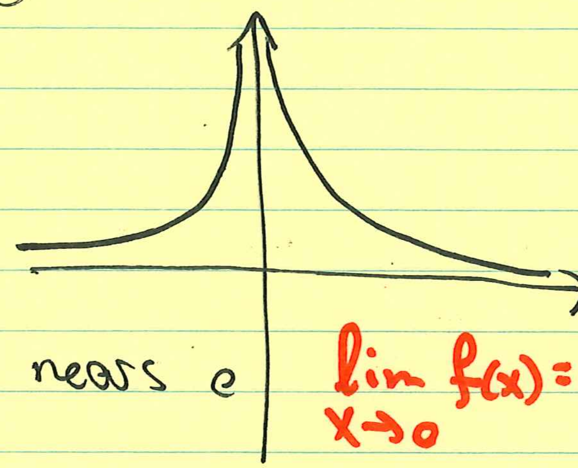
Limits Involving Infinity

Example: $f(x) = \frac{1}{x^2}$



We see that:

① $f(x)$ "explodes" as x nears 0 $\quad \lim_{x\to 0} f(x) = \infty$

② $f(x)$ nears 0 as x grows $\quad \lim_{x\to\infty} f(x) = 0$

Informal definition:

We say that $\lim_{x\to c} f(x) = \infty$ if $f(x)$ is arbitrarily big for x close enough to c.

$x = c$ is called a vertical asymtote of f.

HW: Define:

$$\lim_{x\to c} f(x) = -\infty$$

$$\lim_{x\to c^+} f(x) = \infty \qquad \lim_{x\to c^-} f(x) = \infty$$

$$\lim_{x\to c^+} f(x) = -\infty \qquad \lim_{x\to c^-} f(x) = -\infty$$

Examples:

① $\lim_{x \to 1} \frac{1}{(x-1)^2} = ?$

$$f(\overbrace{1+1}^{2}) = \frac{1}{1^2} = 1$$

$$f(\overset{1.1}{1+0.1}) = \frac{1}{0.1^2} = 100$$

$$f(1+0.1^n) = \frac{1}{(0.1)^{2n}} = 10^{2n}$$

② $f(x) = \frac{1}{x}$ as x nears 0?

$f(1) = 1$	$f(-1) = -1$
$f(0.1) = 10$	$f(-0.1) = -10$
$f(0.01) = 100$	$f(-0.01) = -100$
$\lim_{x \to 0^+} \frac{1}{x} = \infty$	$\lim_{x \to 0^-} f(x) = -\infty$

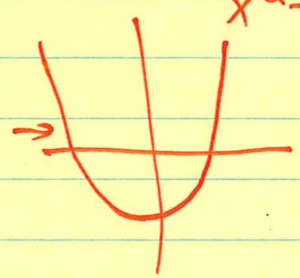
$\lim_{x \to 0} \frac{1}{x}$ doesn't exists.

③ Find vertical asymptotes for

$$f(x) = \frac{3x-6}{x^2-4}$$

options: $+2, -2$.

For $x=-2$: $3x-6 \xrightarrow[x\to -2]{} -12$



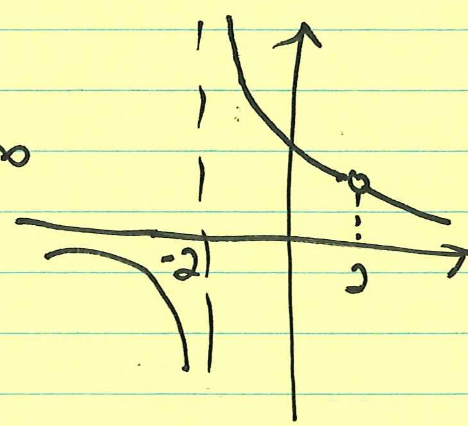
$$x^2-4 \xrightarrow[x\to -2]{} 0$$

$$x^2-4 \xrightarrow[x\to -2^-]{} 0^+$$

$$\frac{3x-6}{x^2-4} \xrightarrow[x\to -2^-]{} -\infty$$

$$x^2-4 \xrightarrow[x\to -2^+]{} 0^-$$

$$\frac{3x-6}{x^2-4} \xrightarrow[x\to -2^+]{} \infty$$



For $x=2$: $3x-6 \xrightarrow[x\to 2]{} 0$

$$\frac{3x-6}{x^2-4} = \frac{3\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \frac{3}{x+2} \xrightarrow[x\to 2]{} \frac{3}{4}.$$

Informal Definition:

We say that $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is arbitrarily close to L when x big enough.

<u>HW</u>: define $\lim_{x\to -\infty} f(x) = L$.

$y = L$ is called a <u>horizontal asymptote</u> of f

Examples:

⓪ $C(q) = 5{,}000 + \frac{1}{2}q$

$$\overline{C(q)} = C_{avg}(q) = \frac{5{,}000}{q} + \frac{1}{2} \xrightarrow[q\to\infty]{} \frac{1}{2}$$

① $$\lim_{x\to\pm\infty} \frac{x^2}{x^2+4} = \lim_{x\to\pm\infty} \frac{1}{1+\underbrace{\frac{4}{x^2}}_{\to 0}} = \frac{1}{1+0} = 1$$

② $\lim_{x\to\infty} \frac{x \;/x}{x^2+1 \;/x} = \lim_{x\to\infty} \frac{1}{\underbrace{x}_{\to\infty} + \underbrace{\frac{1}{x}}_{\to 0}} = 0$

More generally:

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_0} = \begin{cases} 0, & n < m \\ \frac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m \end{cases}$$

③ $\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\pm\infty} \frac{\pm 1}{\sqrt{\frac{x^2+1}{x^2}}} = \lim_{x\to\pm\infty} \frac{\pm 1}{\sqrt{1+\frac{1}{x^2}}} = \pm 1$

④ $\lim_{x\to\infty} \frac{\overbrace{\sin x}^{\text{Bounded}}}{\underbrace{x}_{\to\infty}} = 0$

For any positive integer n

(5) $$\lim_{x \to \infty} \frac{x^n}{e^x} = 0$$

(6) $$\lim_{x \to \infty} \frac{\ln(x)}{x^n} = 0$$

(7) $$\lim_{x \to \infty} \frac{\ln(x)}{e^x} = 0$$

$$= \frac{\ln(x)}{x^n} \cdot \frac{x^n}{e^x}$$

(8) $$\lim_{x \to -\infty} e^x = 0$$

e^x

$\vdots$

x^n

$\vdots$

x^2

x

$\ln(x)$

x^x

e^{e^x}

e^{x^2}