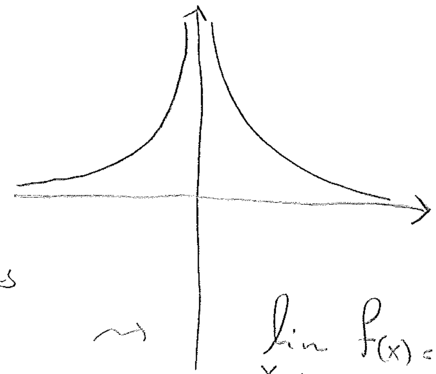
Limits Involving Infinity

Example: $f(x) = \frac{1}{x^2}$



Looking at the graph,

we see that: ① $f(x)$ ~~grows~~ explodes as x nears 0 $\rightsquigarrow$ $\lim\limits_{x \to 0} f(x) = \infty$

② $f(x)$ nears 0 as x grows. $\rightsquigarrow$ $\lim\limits_{x \to \infty} f(x) = 0$

Informal definition:

We say that $\lim\limits_{x \to c} f(x) = \infty$ if $f(x)$ gets arbitrarily bix for x close enough to c.

$x = c$ is called a vertical asymptote of f

HW: Define $\lim\limits_{x \to c} f(x) = -\infty$

$\lim\limits_{x \to c^+} f(x) = \infty$ $\quad$ $\lim\limits_{x \to c^-} f(x) = \infty$

$\lim\limits_{x \to c^+} f(x) = -\infty$ $\quad$ $\lim\limits_{x \to c^-} f(x) = -\infty$

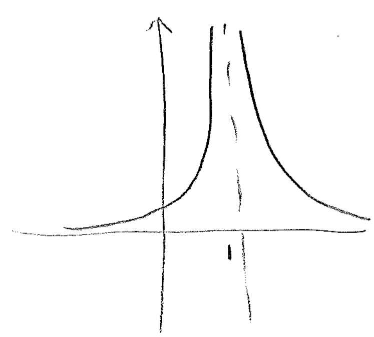
Examples: ① $\lim\limits_{x \to 1} \frac{1}{(x-1)^2} = ?$

$f(\overset{2}{1+1}) = \frac{1}{1^2} = 1$

$f(\overset{1.1}{1+0{,}1}) = \frac{1}{(0{,}1)^2} = 100$

$f(\overset{1.01}{1+0{,}01}) = \frac{1}{(0{,}01)^2} = 10{,}000$

$f(1+10^{-n}) = 10^{2n}$



$\rightsquigarrow \lim\limits_{x \to 1} \frac{1}{(x-1)^2} = \infty$

② $\lim\limits_{x \to 0} \frac{1}{x} = ?$

$f(1) = 1$ $\qquad$ $f(-1) = -1$

$f(0.1) = 10$ $\qquad$ $f(-0.1) = -10$

$f(0.01) = 100$ $\qquad$ $f(-0.01) = -100$

$f(0.001) = 1{,}000$ $\qquad$ $f(-0.001) = -1{,}000$

gets arbitrarily big $\qquad$ gets arbitrarily large (small?) negative values

So, $\lim\limits_{x \to 0} \frac{1}{x}$ doesn't exist.

On the other hand $\lim\limits_{x \to 0^+} \frac{1}{x} = \infty$, $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$.

③ Find vertical asymptotes for $f(x) = \frac{3x-6}{x^2-4}$

Sol: $f(x)$ is not defined for $x = \pm 2$.

For $x = -2$: $3x - 6 \xrightarrow[x \to -2]{} -12$

$x^2 - 4 \xrightarrow[x \to -2]{} 0$ but more importantly $\quad x^2 - 4 \xrightarrow[x \to -2^+]{} 0^-$

$x^2 - 4 \xrightarrow[x \to -2^-]{} 0^+$

So $\lim\limits_{x \to -2^+} f(x) = +\infty$

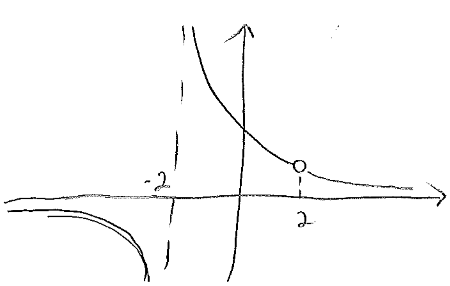
$\lim\limits_{x \to -2^-} f(x) = -\infty$

What about $x = 2$? $\quad 3x - 6 \xrightarrow[x \to 2]{} 0$?

$x^2 - 4 \xrightarrow[x \to 2]{} 0$

$\frac{3x-6}{x^2-4} = \frac{3\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \frac{3}{x+2}$

So $\lim\limits_{x \to 2} f(x) = \frac{3}{2+2} = \frac{3}{4}$



Informal Def: We say $\lim_{x\to\infty} f(x) = L$ if $f(x)$ is arbitrarily close to L as x gets big enough.

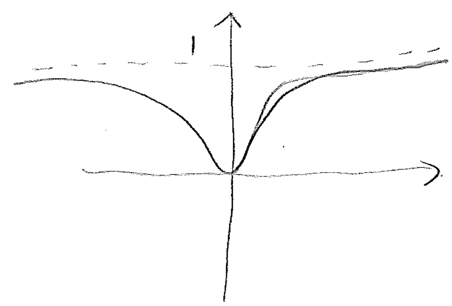
HW: define $\lim_{x\to -\infty} f(x) = L$

$y = L$ is called a horizontal asymptote of f

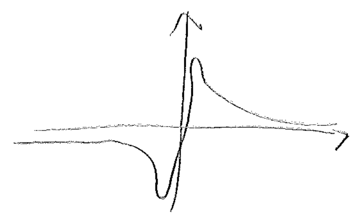
Examples: ⓪ $C(q) = 5{,}000 + \frac{1}{2}q$

$\bar{C}(q) = \frac{5{,}000}{q} + \frac{1}{2} \xrightarrow[q\to\infty]{} \frac{1}{2}$

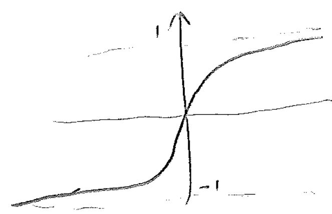
① $\lim_{x\to\pm\infty} \frac{x^2}{x^2+4} = \lim_{x\to\infty} \frac{1}{1+\underbrace{\frac{4}{x^2}}_{\to 0}} = 1$



② $\lim_{x\to\pm\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{1}{\underbrace{x}_{\to\infty} + \underbrace{\frac{1}{x}}_{\to 0}} = 0$

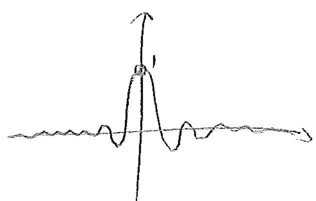


③ $\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\pm\infty} \frac{\pm 1}{\sqrt{1+\frac{1}{x^2}}} = \pm 1$



Idea: $\sqrt{x^2+1} \sim \sqrt{x^2} = |x|$ for x very big

④ $\lim_{x\to\pm\infty} \frac{\overbrace{\sin x}^{\text{bounded}}}{\underbrace{x}_{\to\infty}} = 1$



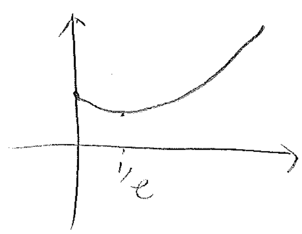
More generally: $\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0} = \begin{cases} 0, & n<m \\ \frac{a_n}{b_m}, & n=m \\ \infty, & n>m \end{cases}$

(5) $\lim_{x \to \infty} \frac{x^n}{e^x} = 0$ for any integer n.

(6) $\lim_{x \to \infty} \frac{\ln(x)}{x^n} = 0$ for any integer n

$\left(\lim_{x \to \infty} \frac{\ln(x)}{e^x} = 0 \right)$

Remark: $x^x \xrightarrow[x \to 0^+]{} 1$



$x^x = e^{x \ln(x)}$

So $x \ln(x) \xrightarrow[x \to 0^+]{} 0$

(8) $\lim_{x \to -\infty} e^x = 0$

T/F: If $f(x) \xrightarrow[x \to \infty]{} \infty$ is it true that $\frac{1}{f(x)} \xrightarrow[x \to \infty]{} 0$?