

Optimization:

① Preparation:

- ① Understand the problem.
- ② Draw a diagram
- ③ Introduce your notations.

② Reduce to one variable

③ Solve the problem - Do calc.

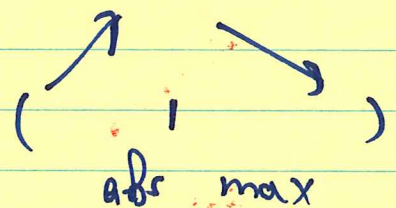
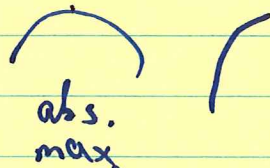
• EVT + closed interval method

• Absolute value method
extreme

• Continuity

• Sketch the graph.

④ Reflect. (& repeat)



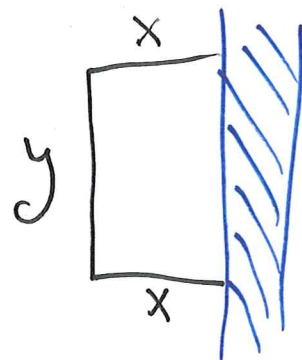
Question 1.

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river. What are the dimensions of the field that has the largest area?

Target function: $A = x \cdot y$

$$2,400 = 2x + y$$

$$\rightarrow y = 2,400 - 2x$$



$$\rightarrow A(x) = x(2,400 - 2x) = -2x^2 + 2,400x$$

$$A'(x) = -4x + 2,400$$

$$A''(x) = -4 < 0 \rightarrow \text{concave down parabola.}$$

\rightarrow There is an absolute maximum.

$$A'(x) = 0 \rightarrow 4x = 2,400 \rightarrow x = 600 \text{ ft.}$$

$$y = 2,400 - 1,200 = 1,200$$

$$A(600) = 600 \cdot 1,200 = 720,000 \text{ ft}^2$$

Question 2.

Find two numbers whose difference is 100 and whose product is a minimum.

x, y - numbers

$$x - y = 100$$

$$P(x) = x \cdot y$$

Target function.

Minimize $P(x)$.

$$y = x - 100$$

$$P(x) = x(x - 100) = x^2 - 100x$$

$$P'(x) = 2x - 100$$

$$P''(x) = 2 > 0 \rightarrow \text{concave up.}$$

\rightarrow There is an abs. min.

$$P'(x) = 0 \rightarrow 2x = 100 \rightarrow \begin{cases} x = 50 \\ y = -50 \end{cases}$$

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\begin{matrix} (x_1, y_1) \\ (x_2, y_2) \end{matrix}$$

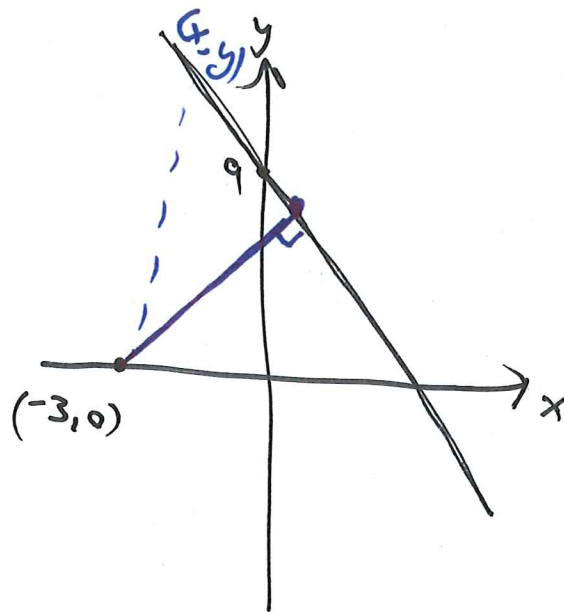
Question 3.

Find the point of the line $6x + y = 9$ that is closest to the point $(-3, 0)$.

~~Handwritten scribbles~~

$$d = \sqrt{(x+3)^2 + y^2}$$

Target function



Aim: Minimized.

$$y = 9 - 6x$$

$$d(x) = \sqrt{(x+3)^2 + (9-6x)^2}$$

* Minimizing $d(x)$ is the same as minimizing $d(x)^2$, since $d(x) > 0$.

$$* f(x) = d(x)^2 = (x+3)^2 + (9-6x)^2$$

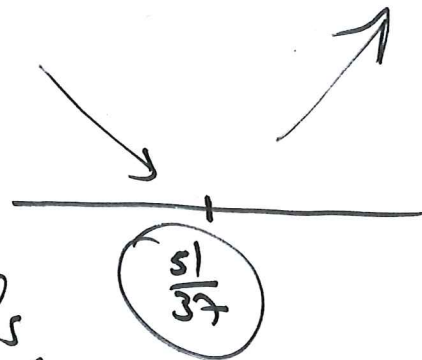
$$f'(x) = 2(x+3) \cdot 1 + 2(9-6x) \cdot (-6)$$

$$= 74x - 102$$

$$f'(x) = 0 \rightarrow x = \frac{51}{37}$$

coordinates of the closest point \rightarrow $\begin{cases} y = 9 - 6 \cdot \frac{51}{37} \end{cases}$

abs. min.



$$d > 0$$

$$f(x) = d(x)^2$$

$$f'(x) = \underbrace{2d(x)}_{>0} \cdot d'(x)$$

$$f'(x) > 0 \quad \text{iff.} \quad d'(x) > 0$$

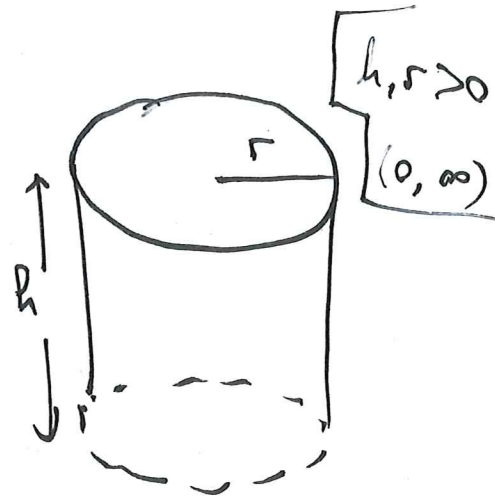
$$f'(x) < 0 \quad \text{iff.} \quad d'(x) < 0$$

$$f'(x) = 0 \quad \text{iff.} \quad d'(x) = 0$$

Question 4.

A cylindrical can is being made to contain 1 L of oil. Find the dimensions that will minimize the amount of metal needed to make the can.

$$A = \underbrace{2 \cdot \pi r^2}_{\text{cap}} + \underbrace{2\pi r \cdot h}_{\text{cylinder}}$$



1000 cm³

$$1 = V = \pi r^2 \cdot h.$$

L.

$$h = \frac{1}{\pi r^2}$$

$$A(r) = 2\pi r^2 + \frac{2\pi r}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

$$A'(r) = 4\pi r - \frac{2}{r^2}$$

$$A'(r) = 0 \quad 4\pi r = \frac{2}{r^2} \rightarrow r^3 = \frac{1}{2\pi} \rightarrow r = \frac{1}{\sqrt[3]{2\pi}}$$

$$A''(r) = 4\pi + \frac{4}{r^3} > 0 \text{ for } r > 0$$

10 cm

→ A is concave up → $\frac{1}{\sqrt[3]{2\pi}}$ is the global minimum of A(r) in (0, ∞).

$$h = \frac{1}{\pi r^2}$$

$$L = 1000 \text{ cm}^3$$

$$l = (10 \text{ cm})^3$$

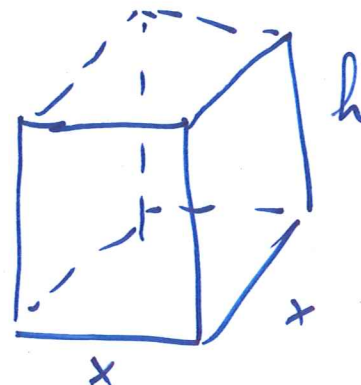
$$r = \frac{10}{\sqrt[3]{2\pi}} \text{ cm}$$

Question 5.

If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.

$$V = x^2 \cdot h \quad \leftarrow \text{Target function.}$$

$$x, h > 0$$

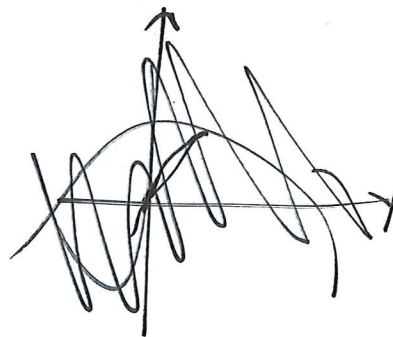


$$\underbrace{x^2}_{\text{Bottom}} + \underbrace{4 \cdot x \cdot h}_{\text{Side}} = 1,200 \text{ cm}^2$$

$$h = \frac{1200 - x^2}{4x}$$

$$V(x) = x^2 \cdot \left(\frac{1200 - x^2}{4x} \right) = \frac{1200}{4} x - \frac{x^3}{4}$$

$$= 300x - \frac{1}{4}x^3 \quad x > 0$$



$$V' = 300 - \frac{3}{4}x^2 \stackrel{!}{=} 0$$

$$x^2 = 400$$

$$x = \pm\sqrt{400} = \pm 20$$

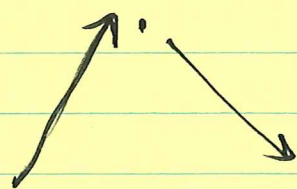
$x = 20$. only critical point

for $x > 0$.

$(0, \infty)$

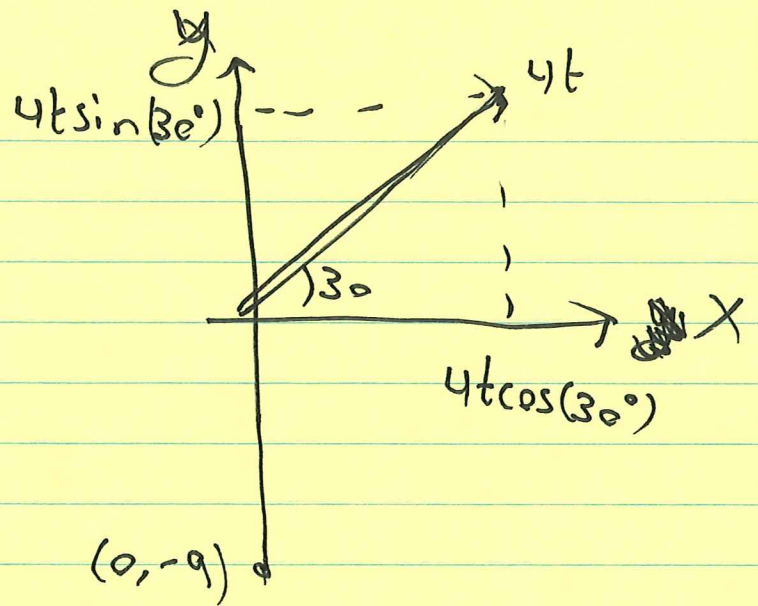
~~$V'(x) > 0$~~ $V'(x) > 0 \quad x < 20$

$V'(x) < 0 \quad x > 20$



$x = 20$ is the global maximum
on $(0, \infty)$

$$V(20) = 300 \cdot 20 - \frac{1}{4} 20^3$$



$$d(t) = \sqrt{(4t \cos(30^\circ))^2 + (4t \sin(30^\circ) + 9)^2}$$

Question 6.

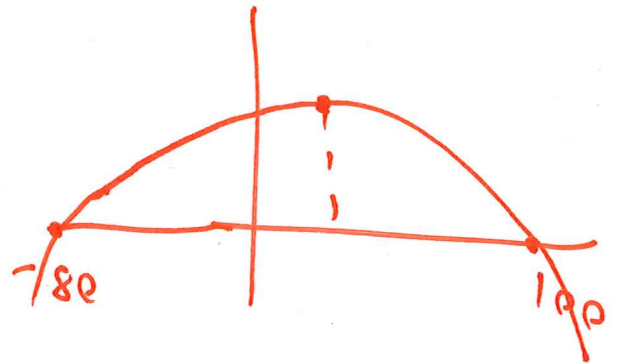
The manager of a 100-unit apartment complex knows from experience that all the units will be occupied if the rent is \$800 per month. A market survey indicates that one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

$x = \#$ of \$10 increases

$$\begin{cases} P = 800 + 10x \\ Q = 100 - x \end{cases}$$

$$R = P \cdot Q = (800 + 10x)(100 - x)$$

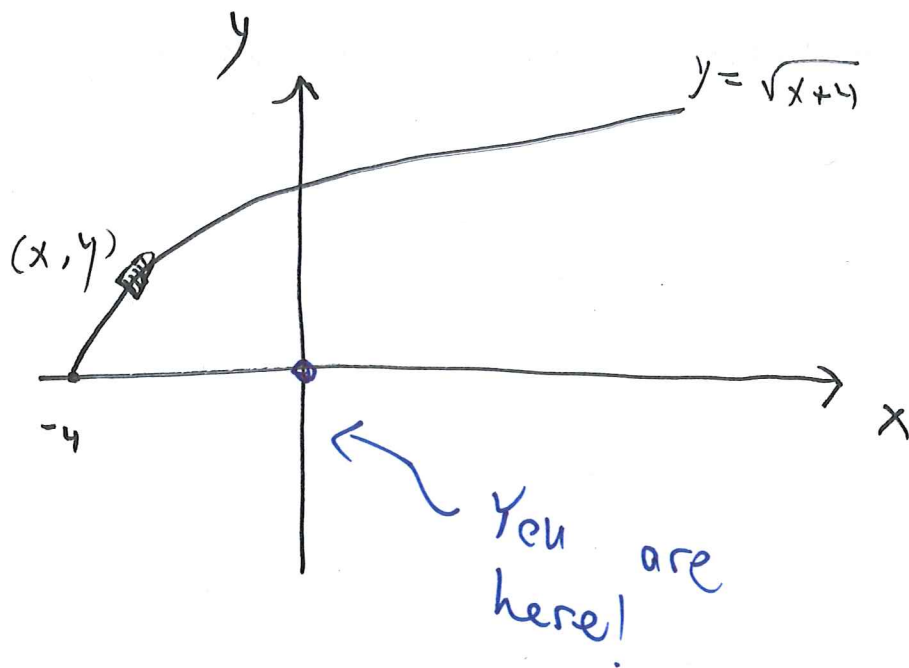
$$x_{\max} = \frac{100 + (-80)}{2} = 10$$



$$P = 800 + 10 \cdot 10 = \$900$$

Question 7.

You stand on a cliff at point $(0,0)$ overlooking a river. You see a boat due north at point $(0,2)$. The boat is traveling down the river along the curve $y = \sqrt{x+4}$ towards the harbour at $(-4,0)$. You want to wave to the boat at the point where it is closest to you. Find the coordinates of this point.



$$d = \sqrt{y^2 + x^2} \quad \text{target function}$$

Goal: minimize $d(x)$

$$y = \sqrt{x+4}$$

$$d = \sqrt{\sqrt{x+4}^2 + x^2} = \sqrt{x^2 + x + 4}$$

$$d'(x) = \frac{1}{2\sqrt{x^2 + x + 4}} (2x + 1)$$

0

Critical points:

$$* d'(x) = 0 \longrightarrow x = -\frac{1}{2}.$$

When is ~~when~~ $d'(x)$ defined?

$$x^2 + x + 4 > 0$$

(for any x)

~~$$\frac{-1 \pm \sqrt{1 - 16}}{2}$$~~

$$d'(x) > 0 \quad x > -\frac{1}{2}$$

$$d'(x) < 0 \quad x < -\frac{1}{2}$$

$$\longrightarrow x = -\frac{1}{2}$$

is a global

minimum.

$$y = \sqrt{-\frac{1}{2} + 4}$$

$$= \sqrt{\frac{7}{2}}$$

Question 8.

A tutoring company is offering a workshop for the upcoming exam in December. Market research suggests that setting the price of the solution at p in dollars will yield

$$q(p) = 20(18 - 2\sqrt{p})$$

students registering. What price should they set in order to maximize revenue?

Option I:

$$R(p) = p \cdot q(p) = 20p(18 - 2\sqrt{p}) = 20(18p - 2p^{3/2})$$

$$R'(p) = 20(18 - 3\sqrt{p}) \stackrel{!}{=} 0 \rightarrow \sqrt{p} = 6 \rightarrow p = 36$$

$R' > 0$ 36 $R' < 0$

Global max

$\sqrt{p} = 6 \rightarrow p = 36$

Option II:

$$\frac{dq}{dp} = -\frac{20}{\sqrt{p}}$$

$$E_d = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{-20\sqrt{p}}{20(18 - 2\sqrt{p})} = \frac{-\sqrt{p}}{18 - 2\sqrt{p}} \stackrel{!}{=} -1$$

$$\rightarrow \sqrt{p} = 18 - 2\sqrt{p}$$

$$\rightarrow 3\sqrt{p} = 18$$

$$\rightarrow \sqrt{p} = 6$$

$$p = 36$$

Question 9. You are planning a city tour for a group of 100 tourists. If you can sell x bus tour tickets, you can offer them for $\$30 - x/4$ each. If you can sell y train tour tickets, you can offer them for $\$70 - y/2$ each. How many bus tickets, and how many train tickets should you sell to the tourists in order to maximize revenue (you can only sell one type of ticket to each passenger).

$$x + y = 100 \rightarrow y = 100 - x$$

$$R = x\left(30 - \frac{x}{4}\right) + y\left(70 - \frac{y}{2}\right)$$

Target
function

max it

$$R(x) = x\left(30 - \frac{x}{4}\right) + (100 - x)\left(70 - \frac{100 - x}{2}\right)$$

$$= -\frac{3}{4}x^2 + 60x + 2000$$

← R is concave
down

$$R'(x) = -\frac{3}{2}x + 60 \stackrel{!}{=} 0$$

→ going to have
a global max.

$$\boxed{x=40, y=60}$$

Question 10. A steel company, ABC steel, manufactures nuts and bolts. When x nuts are produced, they can be sold for $-3x + 500$ dollars each. When y bolts are produced, they can be sold for $-y + 300$ dollars each.

Assume that nuts and bolts weigh 0.5kg each. How many nuts and how many bolts must be produced to maximize the revenue from 100kg of steel? Justify your answer.

as x, y integers ≤ 200

$$100 = \frac{1}{2}x + \frac{1}{2}y$$

$$y = 200 - x$$

$$R = x(-3x + 500) + y(-y + 300)$$

$$= x(500 - 3x) + (200 - x)(100 + x)$$

$$= -3x^2 + 500x + 20000 - 100x + 200x - x^2$$

$$= -4x^2 + 600x + 20,000$$

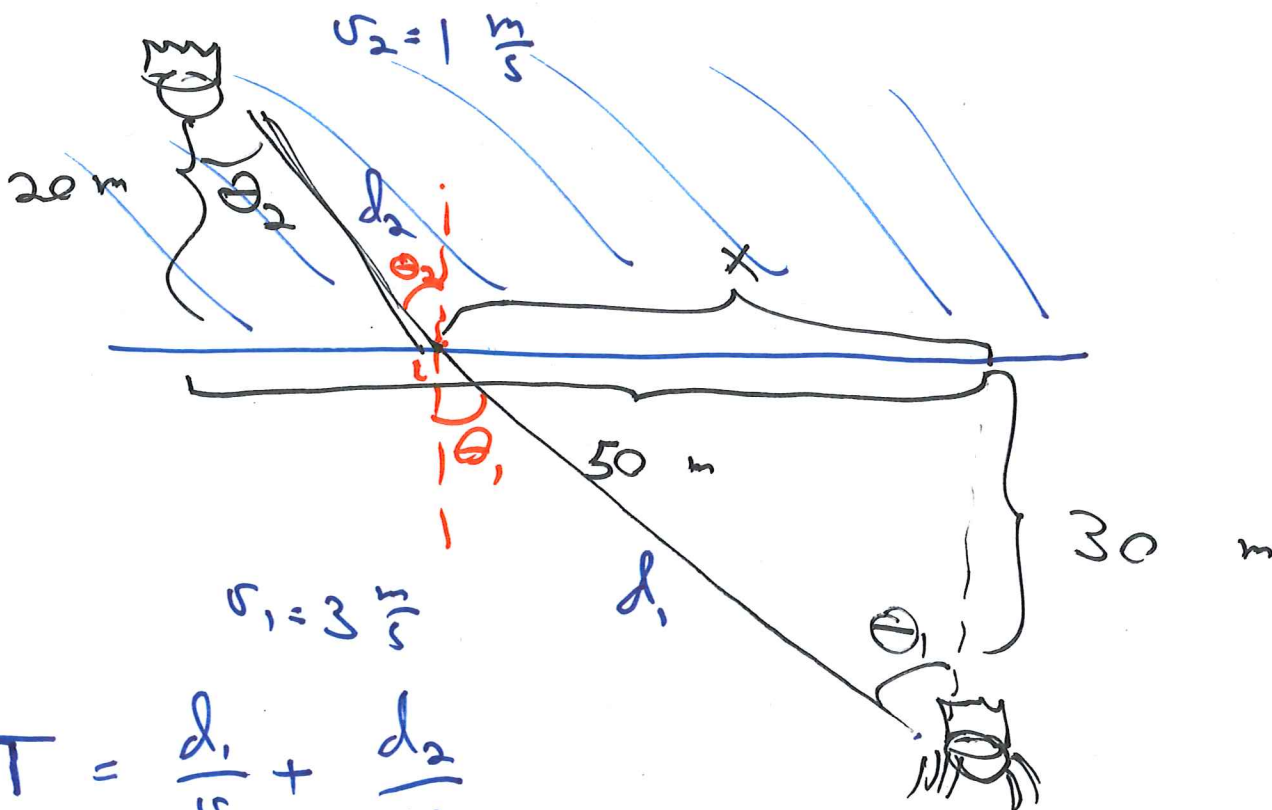
$$R'(x) = -8x + 600 = 0$$

$$x = \frac{600}{8} = 3 \cdot 25 = 75$$

$$y = 200 - 75 = 125$$

$$R = 75 \cdot 175 + 125 \cdot 175$$

Question 11. A prince is drowning 20 meters off a straight coast. A princess sees him and wants to save him. She is standing 30 meters off the coast line and 50 meters to the right. She can run at the speed of 3 meters/second and swim at the speed of 1 meter/second. What is the course in which she should run and swim to get to the prince as fast as she can?



$$T = \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

$$d_1 = \sqrt{x^2 + 30^2}$$

$$d_2 = \sqrt{(50-x)^2 + 20^2}$$

$$T(x) = \frac{\sqrt{x^2 + 30^2}}{3} + \frac{\sqrt{(50-x)^2 + 20^2}}{1}$$

$$T'(x) = \frac{2x}{3 \cdot 2 \sqrt{x^2 + 30^2}} + \frac{-2(50-x)}{1 \cdot 2 \sqrt{(50-x)^2 + 20^2}}$$

$$= \frac{x}{3 \sqrt{x^2 + 30^2}} - \frac{(50-x)}{1 \sqrt{(50-x)^2 + 20^2}}$$

$$= \frac{x}{3 \sqrt{x^2 + 30^2}} - \frac{50-x}{1 \sqrt{(50-x)^2 + 20^2}}$$

$$= \frac{\frac{x}{dx}}{3} - \frac{\frac{50-x}{dx}}{1} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}$$

$$T'(x) = 0 \rightarrow \boxed{\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}} \quad \begin{array}{l} \text{Snell's} \\ \text{Law} \end{array}$$