Approximations

Problem 1.
Approximate $\sqrt{73}$ (as a simple or decimal fraction) without using a calculator.

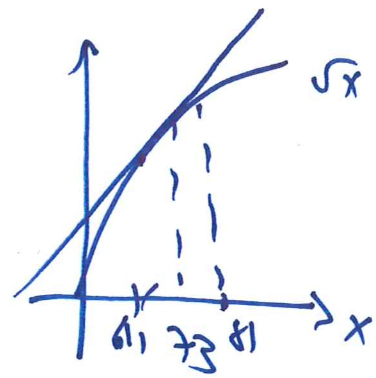
- The closest squares to 73 are

$$64 < 73 < 81$$

So $$8 = \sqrt{64} < \sqrt{73} < \sqrt{81} = 9$$

$$l_1(x) = f(64) + f'(64)(x-64)$$

$$l_2(x) = f(81) + f'(81)(x-81)$$



$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(64) = \frac{1}{2\cdot 8} = \frac{1}{16}$$

$$f'(81) = \frac{1}{2\cdot 9} = \frac{1}{18}$$

$$l_1(x) = 8 + \frac{1}{16}(x-64) \qquad l_1(73) = \frac{137}{16} = 8.56..$$

$$l_2(x) = 9 + \frac{1}{18}(x-81) \qquad l_2(73) = \frac{77}{9} = 8.55..$$

$$\sqrt{73} \approx 8.54...$$

Why does it work?

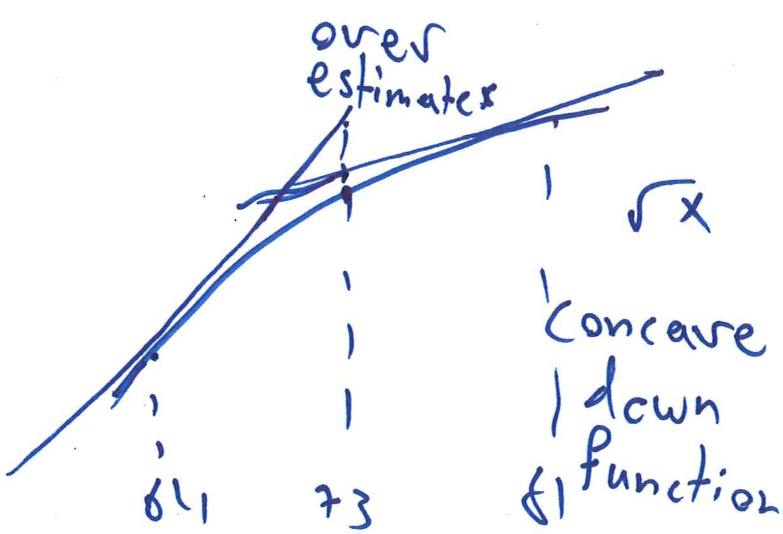
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\left(\begin{array}{l} x \neq a \\ x \text{ near to } a \end{array} \right)$$

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

approx.

$$f(a) + f'(a)(x-a) \approx f(x)$$



Concave down → over estimates

Concave up → under estimates.

Problem 2.
Approximate ln(2) (as a simple or decimal fraction) without using a calculator.

- $f(x) = \ln(x)$
- We know that $f(1) = \ln(1) = 0$

$$f(e) = \ln(e) = 1$$

and $\quad 1 < 2 < e$

- $f'(x) = \frac{1}{x}$

we don't have an approx. for $\frac{1}{e}$
So we use only $a = 1$

$$l(x) = \underbrace{f(1)}_{0} + \underbrace{f'(1)}_{1} \cdot (x-1) = x-1$$

$l(2) = 2 - 1 = 1.$

$\ln(2) \approx 0.69\ldots$

Let's try to use a quadratic polynomial.

$$P(x) = A + Bx + Cx^2$$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$\begin{cases} P(1) = f(1) \\ P'(1) = f'(1) \\ P''(1) = f''(1) \end{cases}$$

$$P'(x) = B + 2Cx$$

$$P''(x) = 2C$$

$$\begin{cases} A + B + C = 0 \\ B + 2C = 1 \\ 2C = -1 \end{cases}$$

$$C = -\frac{1}{2}$$

$$B = 1 - 2c = 2$$

$$A = -B - C = -\frac{3}{2}$$

$$P(x) = -\frac{3}{2} + 2x - \frac{1}{2}x^2$$

$$P(2) = -\frac{3}{2} + 4 - 2 = \frac{1}{2} = 0.5$$

$P_1(x) = l(x) = A + B(x-1) = f(a) + f'(a)(x-1)$

$P_2(x) = A + B(x-1) + C(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$

$P_2(1) = f(1) = 0$

$P_2'(1) = f'(1) = 1$

$P_2''(1) = f''(1) = -1$

$P_2(1) = A$

$P_2'(x) = B + 2C(x-1)$

$P_2'(1) = B$

$P_2''(x) = 2C$

$A = 0$

$B = 1$

$2C = -1 \qquad C = -\frac{1}{2}$

$P_3(x) = A + B(x-1) + C(x-1)^2 + D(x-1)^3$

$A = P_3(1) = f(1) = 0$

$B = P_3'(1) = f'(1) = 1$

$2C = P_3''(1) = f''(1) = -1$

$D = \frac{1}{3} \Leftarrow 6D = P_3^{(3)}(1) = f^{(3)}(1) = 2$

$f''(x) = -\frac{1}{x^2}$

$f^{(3)}(x) = \frac{2}{x^3}$

$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

$$P_3'(x) = B + 2C(x-1) + 3D(x-1)^2$$

$$P_3''(x) = 2C + 6D(x-1)$$

$$P_3^{(3)}(x) = 6D$$

$$P_3(2) = (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3$$

$$= 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \approx 0.83\ldots$$

Let's say we want to approx. $f(x)$ by an n-th degree polynomial around a.

$$\boxed{P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_{n-1}(x-a)^{n-1} + a_n(x-a)^n = \sum_{k=0}^{n} a_k(x-a)^k}$$

n-th Taylor Polynomial

where

$$a_0 = f(a),\ a_1 = f'(a),\ a_2 = \frac{f''(a)}{2},\ a_3 = \frac{f^{(3)}(a)}{6},$$

$$a_4 = \frac{f^{(4)}(a)}{24},\ a_5 = \frac{f^{(5)}(a)}{120}$$

$$\boxed{a_k = \frac{f^{(k)}(a)}{k!}}$$

$2,\ 6 = 2\cdot 3,\ 24 = 2\cdot 3\cdot 4,\ 120 = 2\cdot 3\cdot 4\cdot 5 = 5!$

$$k! = k(k-1)\cdots 3\cdot 2\cdot 1$$

Taylor polynomials:

For a function $f(x)$, the n-th <u>Taylor polynomial</u> of $f(x)$ centered at $x=a$ is

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$$

$$a_k = \frac{1}{k!} f^{(k)}(a)$$

If $a=0$ then $P_n(x)$ is called the <u>Maclaurin polynomial</u>.

$\sin x \approx x$

Problem 2'.

Recall Last time we computed the following three Taylor polynomials of $f(x) = \ln x$ around $x = 1$

$$p_1(x) = (x-1)$$
$$p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$
$$p_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

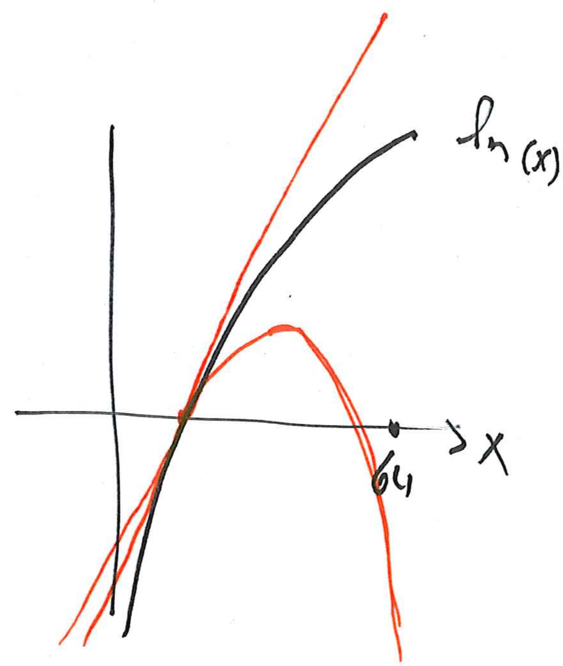
Approximate $\ln(64)$ (as a simple or decimal fraction) without using a calculator.

$$P_1(64) = 63$$

$$P_2(64) = 63 - \frac{1}{2}(63)^2 = -1921.5$$

$$P_3(64) = 63 - \frac{1}{2}(63)^2 + \frac{1}{3}(63)^3 = 81427.5$$

$$\ln(64) \approx 4.15\ldots$$



$$64 = 8^2$$
$$\ln(64) = 2\ln(8)$$
$$64 = 2^6$$
$$\ln(64) = 6\ln(2)$$

$$6 \cdot P_3(2) = 6\left(1 - \frac{1}{2} + \frac{1}{3}\right) = 6 \cdot \frac{5}{6} = 5$$

Problem 3.
Given the demand equation $(p+10)(q+20) = p^2q$, find the 2nd Taylor polynomial of q as a function of p around $(5, 30)$.

Derive w.r.t. p:

$$(*) \quad (q+20) + (p+10)\,q' = 2p\cdot q + p^2 q'$$

plug in $p=5$, $q=30$ and solve for $q'(5)$

$$50 + 15\cdot q'(5) = 10\cdot 30 + 25\cdot q'(5)$$

$$q'(5) = -25.$$

Deriving $(*)$ w.r.t. p:

$$q' + 1\cdot q' + (p+10)\,q'' = 2q + 2p\cdot q' + 2p\cdot q' + p^2 q''$$

plug in $p=5$, $q=30$, $q'(5) = -25$

$$q''(5) = 39.$$

$$\boxed{Q_2(p) = \underbrace{30}_{\frac{1}{1}\cdot q(5)} + \underbrace{-25}_{\frac{1}{1}q'(5)}(p-5) + \underbrace{\frac{39}{2}}_{\frac{1}{2}\cdot q''(5)}(p-5)^2}$$

$\frac{1}{1}\cdot q(5)$ = 30

$0! = 1 \quad 1! = 1 \quad 2! = 2 \quad 3! = 6 \quad 4! = 24 \quad 5! = 120$

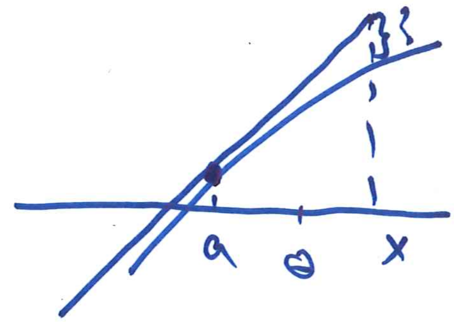
Error Term for Taylor Polynomials

Error Term for Linear Approximations

$$P_1(x) = l(x) = f(a) + f'(a)(x-a)$$

Theorem (Lagrange)

$$f(x) - l(x) = \frac{1}{2} f''(\theta) \cdot (x-a)^2$$



Where θ is between x and a.

Conclusion
~~Corollary~~:

$$|f(x) - l(x)| = \frac{(x-a)^2}{2} |f''(\theta)|$$

$$\leq \frac{(x-a)^2}{2} \cdot M$$

Where:

$$M = \text{Max } |f''(\theta)|$$

for θ between a and x.

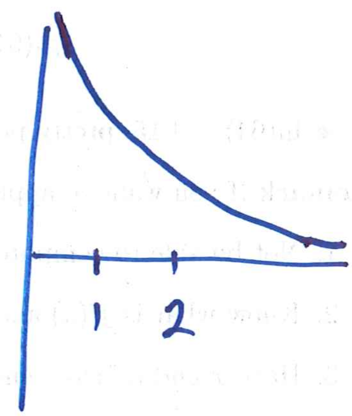
Example: $f(x) = \ln x$, $a = 1$ $(x = 2)$

$$l(x) = x - 1$$

$$f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2}$$

$$|f''(x)| = \frac{1}{x^2}$$

is a decreasing function on $[1, 2]$



$$M = \max_{\substack{x \text{ between} \\ 1 \text{ and } 2}} |f''(x)| = |f''(1)| = 1$$

and hence:

$$|f(2) - l(2)| \leq \frac{(2-1)^2}{2} \cdot M = \frac{1}{2}$$

$l(2) = 1$, $f(2) = 0.69\ldots$

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Example: $f(x)=\sqrt{x}$, $a_1=64$, $a_2=81$ $(x=73)$

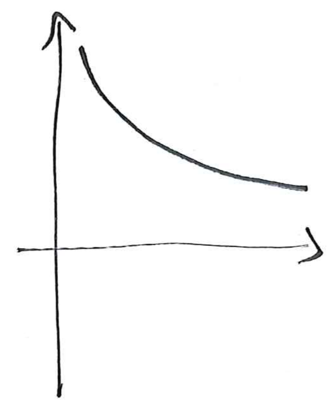
$(x^r)' = r x^{r-1}$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}} = \frac{-1}{4\sqrt{x^3}}$$

$$|f''(x)| = \frac{1}{4\sqrt{x^3}}$$

decreasing function.



$$M_1 = \max_{64 \le x \le 73} |f''(x)| = |f''(64)|$$

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$$M_2 = \max_{73 \le x \le 81} |f''(x)| = |f''(73)|$$

$$\frac{(64-73)^2}{2} \cdot M_1 > \frac{(81-73)^2}{2} M_2$$