Linear approximations:

- $f(x)$
- The linear approximation of $f(x)$ centered at a is

$$l(x) = f(a) + f'(a)\,(x-a).$$

- $$|f(x) - l(x)| \le \frac{|x-a|^2}{2} M$$

where $M = \max |f''(z)|$, z between x and a

- f concave down $\longrightarrow$ over estimate

f concave up $\longrightarrow$ under estimate.

Taylor polynomials:

$$T_n(x) = P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2$$

$$+ a_3(x-a)^3 + \ldots + a_n(x-a)^n$$

$$a_0 = f(a), \quad a_1 = f'(a), \quad a_2 = \frac{f''(a)}{2}, \quad a_3 = \frac{f'''(a)}{6}$$

$$a_k = \frac{f^{(k)}(a)}{k!}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots + \frac{x^n}{n!}$$

(Centered at 0 – Maclauren poly.)

$\cos x$

$\sin x$

Example: Aprox. $\sqrt{73}$

$a = 64$ $\qquad$ $a = 81$

$l_1(x) = 8 + \frac{1}{16}(x-64)$ $\qquad$ $l_2(x) = 9 + \frac{1}{18}(x-81)$

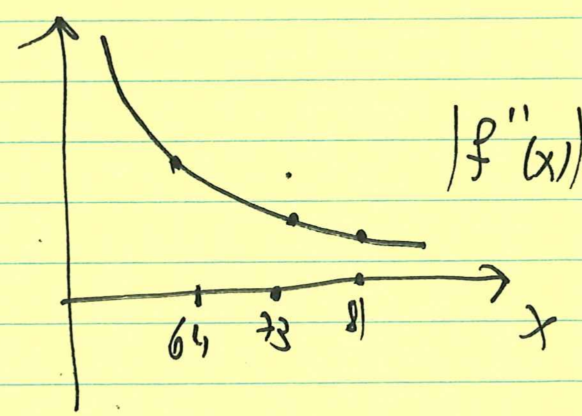
$l_1(73) = \frac{137}{16} \approx 8.562\ldots$ $\qquad$ $l_2(73) = \frac{77}{9} \approx 8.555$

M_1 $\qquad$ M_2

$\sqrt{73} \approx 8.544\ldots$

$$\left[\ |f(x) - l(x)| \le \frac{(x-a)^2}{2} M\ ;\ M = \max_{\substack{z \text{ between} \\ a \text{ and } x}} |f''(z)|\ \right]$$

$$|f''(x)| = \left|\frac{-1}{4x^{3/2}}\right| = \frac{1}{4x^{3/2}}$$

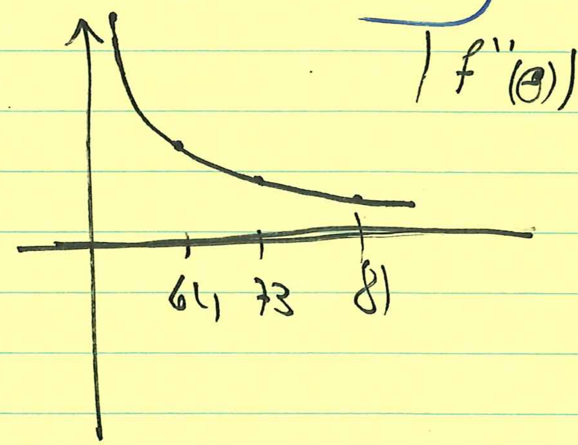


$$M_1 = \frac{1}{4(64)^{3/2}} = \quad > \quad M_2 = \frac{1}{4\cdot(73)^{3/2}}$$

$$|\sqrt{73} - l(73)| < \frac{(73-64)^2}{2} M_1 \qquad |\sqrt{73} - l_2(73)| \le \frac{(81-73)^2}{2} \cdot M_2$$

$$|f(x) - l(x)| = \frac{(x-a)^2}{2} \cdot |f''(\theta)|$$

for some θ between a and x.



$$(x-a)^2 |f''(\theta)|$$

$$(73-64)^2 |f''(\theta_1)| \quad 64 \le \theta_1 \le 73$$

$$(81-73)^2 |f''(\theta_2)| \quad 73 \le \theta_2 \le 81$$

Since $(73-64)^2 > (81-73)^2$

and $|f''(\theta_1)| > |f''(\theta_2)|$ for any

$64 \le \theta_1 \le 73$ & $73 \le \theta_2 \le 81$

So the error for $l_2(73)$ is smaller!

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Problem 4.
Find the n-th Maclauren polynomial of $f(x) = e^x$ and approximate e up to the 2-nd decimal place.

Recall:

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$
$$+ a_n(x-a)^n$$

$$a_k = \frac{1}{k!} f^{(k)}(a)$$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$
$$+ \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $f(x) = e^x$

$f''(x) = f'(x) = e^x$

for any k: $f^{(k)}(x) = e^x$.

$a = 0$: $\quad a_k = \dfrac{f^{(k)}(0)}{k!} = \dfrac{1}{k!}$

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!}.$$

$f(x) = e^x$

$f(1) = e^1 = e$

$P_n(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots + \frac{1}{n!}$

$P_1(1) = 2$

$P_2(1) = 1 + 1 + \frac{1}{2} = 2\frac{1}{2}$

$P_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3} = 2\frac{5}{6}$

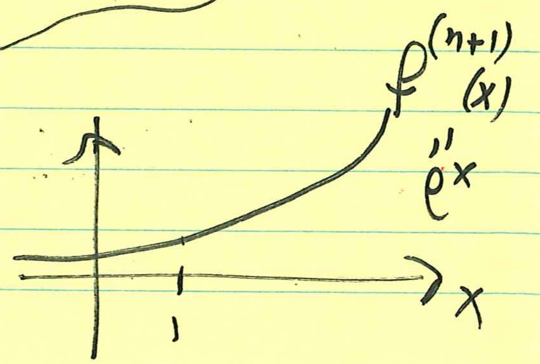
$\vdots$

we want:

$$|f(1) - P_n(1)| < \frac{1}{100}$$

<u>Fact:</u> $|f(1) - P_n(1)| \le \frac{|x-a|^{n+1}}{(n+1)!}$ $\quad M = \frac{e}{(n+1)!} < \frac{4}{(n+1)!}$ $\quad e<4$

$$M = \underset{\substack{x \text{ between} \\ 0 \text{ and } 1}}{\text{Max}} \left| f^{(n+1)}(x) \right| = e$$



$\frac{4}{(n+1)!} < \frac{1}{100}$ $\quad \boxed{n = 5}$

Problem 5.
Find the n-th Maclauren polynomial of $f(x) = \cos x$ and $g(x) = \sin x$.

$a = 0$

$Cos(0) = 1$

$Cos'(x) = -Sin(x)$

$cos'(0) = 0$

$Cos''(x) = -Cos(x)$

$Cos''(x) = -1$

$Cos'''(x) = Sin(x)$

$Cos'''(0) = 0$

$Cos^{(4)}(x) = Cos(x)$

$Sin(0) = 0$

$Sin'(x) = Cos(x)$

$Sin'(0) = 1$

$Sin''(x) = -Sin(x)$

$Sin''(0) = 0$

$Sin'''(x) = -Cos(x)$

$Sin'''(0) = -1$

$Sin^{(4)}(x) = Sin x$

$C_0(x) = 1$

$C_1(x) = 1$

$C_3(x) = C_2(x) = 1 - \frac{1}{2}(x-0)^2$

$= 1 - \frac{x^2}{2}$

$C_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$

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$S_0(x) = 0$

$S_2(x) = S_1(x) = x$

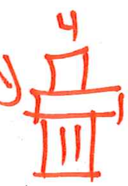
$S_4(x) = S_3(x) = x - \frac{1}{6}x^3$

$S_6(x) = S_5(x) = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5$

Problem 6.

Find the 4-th Maclauren polynomial of $f(x) = e^{x^2}$.

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} +$$  4th Mac. poly. of e^x

$$P_4(x^2) = 1 + x^2 + \frac{x^4}{2} + \boxed{\frac{x^6}{6} + \frac{x^8}{24}} +$$ 

$$= 1 + x^2 + \frac{x^4}{2}$$

Example:

The 4-th Mac. poly. for

$$f(x) = e^x \cdot \cos(x).$$

$$e^x \sim 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \overset{4}{\boxed{|||}}$$

$$\cos(x) \sim 1 - \frac{x^2}{2} + \frac{x^4}{24} + \overset{4}{\boxed{|||}}$$

$$e^x \cdot \cos(x) \approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \boxed{4}\right)$$

$$\cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \boxed{4}\right)$$

$$= 1 + x + \cancel{\frac{1}{2}x^2} - \cancel{\frac{1}{2}x^2} + 1 \cdot \frac{x^3}{6} - \frac{1}{2}x^3$$

$$+ \frac{x^4}{24} - \frac{1}{4}x^4 + \frac{1}{24}x^4 + \boxed{4}$$

$$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4$$

$$\frac{1}{12} - \frac{1}{4}$$

$$= \frac{1-3}{12} = -\frac{2}{12}$$

$$f(x) = e^x \cdot \cos(x)$$

$$P_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$a_0 = f(0) = e^0 \cdot \cos(0)$$

$$a_1 = f'(0)$$

$$f'(x) = e^x \cdot \cos(x) + e^x(-\sin(x))$$

$$= e^0 \cdot \cos(0) + e^0(-\sin(0))$$

$$a_2 = \frac{f''(0)}{2}$$

$$f''(x) = e^x \cdot \cos(x) - 2e^x \sin(x) - e^x \cdot \cos(x)$$