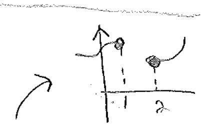
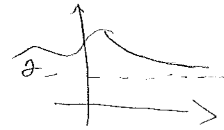
Limit Rules or How to Compute Limits: W2-F

Recall:

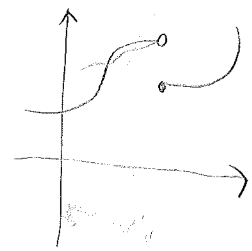




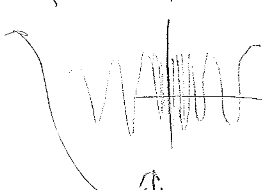
Suppose that $f(x)$ is defined for all x near some point $x=a$. If $f(x)$ is arbitrarely close to some (fixed) value L for all x close to (but not including) $x=a$ then we write

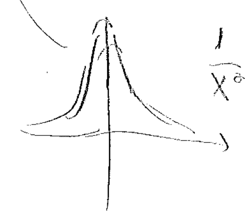
$$\boxed{\lim_{x\to a} f(x) = L}$$

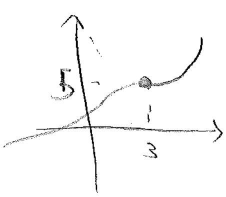
also $f(x) \xrightarrow[x\to a]{} L$.

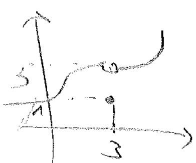


- "Fixed value": There exist at most one limit
- "But not including": The value of the function at $x=a$ is irrelevant to the limit.









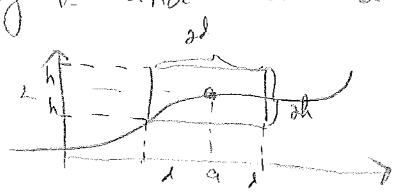
Moreover:

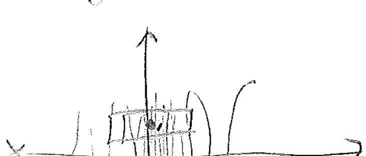
Theorem: If we have $f(x)=g(x)$ for all x near (but not equal to) a then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Example: $\lim_{x\to 1} \frac{x^3-x}{x-1}$? " $\frac{1^3-1}{1-1} = \frac{0}{0}$? "

Note that for $x \neq 1$ $\frac{x^3-x}{x-1} = x(x+1)$ and that $\lim_{x\to 1} x(x+1) = 2 \Rightarrow \lim_{x\to 1} \frac{x^3-x}{x-1} = 2$

- "arbitrarely close to ... for all x close to a": we need to be able to "fit the graph into a rectangle" you give me the height and need to supply the width





Theorem (Basic Limit properties):

Let b, c, L and K be real numbers, let n be a positive integer and let f and g be the functions with the following limits: $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

The following limits hold:

① Constants: $\lim_{x \to c} b = b$

② Identity: $\lim_{x \to c} x = c$ ("x appr. c as x appr. c")

③ Sums/Differences: $\lim_{x \to c} \left(f(x) \pm g(x)\right) = L \pm K$

④ Scalar Multiples: $\lim_{x \to c} b f(x) = b \cdot L$

⑤ Products: $\lim_{x \to c} f(x) \cdot g(x) = L \cdot K$

⑥ Quotients $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ when $K \neq 0$

⑦ powers: $\lim_{x \to c} f(x)^n = L^n$

⑧ Roots: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ if $L > 0$ or n odd

⑨ Compositions: Now assume $\lim_{x \to c} f(x) = L$, $\lim_{x \to L} g(x) = K$ and $g(L) = K$

Then: $\lim_{x \to c} g(f(x)) = K$

Example: $\lim_{x\to 3}(x^2+2x+1) = \lim_{x\to 3} x^2 + \lim_{x\to 3} 2x + \lim_{x\to 3} 1$

$$= \left(\lim_{x\to 3} x\right)^2 + 2\lim_{x\to 3} x + 1 = (3)^2 + 2\cdot 3 + 1 = 16$$

Example: $\lim_{x\to 2} \sqrt[3]{4x^2+3x+5} = \sqrt[3]{\lim_{x\to 2}(4x^2+3x+5)} =$

$$= \sqrt[3]{4\cdot 2^2+3\cdot 2+5} = \sqrt[3]{27} = 3.$$

Question: When $\lim_{x\to a} f(x) = f(a)$?

Infact, if this happen we say that f is continuous at a.

Examples of Cont. func.: polynomials, $\sqrt[m]{x}$ for $x \geq 0$, e^x, $\ln(x)$ for $x>0$, $\cos(x)$, $\sin(x)$,

Also: additions, substractions, multiplications, divisions* and compositions* of cont. functions are continuous.

* while taking care not to divide by zero and that the composition satisfy the assumptions of ①.

Another examples :

$$\frac{1+1-2}{1+4-5} = \frac{0}{0} ?$$

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2+4x-5} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x+5)} = \lim_{x\to 1} \frac{x+2}{x+5} = \frac{3}{5}$$

Theorem : Let $p(x)$ be a polynomial. Then $P(a)=0$ if and only if $(x-a)$ is a factor of $P(x)$.
That is, there is another polynomial, say $Q(x)$, such that
$$P(x) = (x-a)\,Q(x).$$

In this case, there exist a positive integer n and a polynomial $R(x)$ such that $P(x) = (x-a)^n R(x)$ and $R(a) \neq 0$.

Let's see another examples:

$$\lim_{x\to 3} \frac{x^2-2x-3}{2x^2-3x-9} = ?$$

$3^2-2\cdot 3-3=0$, $2\cdot 3^2-3\cdot 3-9=0$ so $(x-3)$ is a factor of

Both x^2-2x-3 and $2x^2-3x-9$.

$$x^2-2x-3 = (x-3)(x+1)$$

$$\begin{array}{r|l} & 2x+3 \\ \hline x-3 & 2x^2-3x-9 \\ & -(2x^2-6x) \\ \hline & 3x-9 \\ & 3x-9 \\ \hline & 0 \end{array}$$

$(x-3)(2x+3) = 2x^2-3x-9$

($2\cdot 3+3 \neq 0$ and hence $n=1$ here)

$$\lim_{x\to 3} \frac{x^2-2x-3}{2x^2-3x-9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}(2x+3)} = \lim_{x\to 3} \frac{x+1}{2x+3} = \frac{4}{9}$$