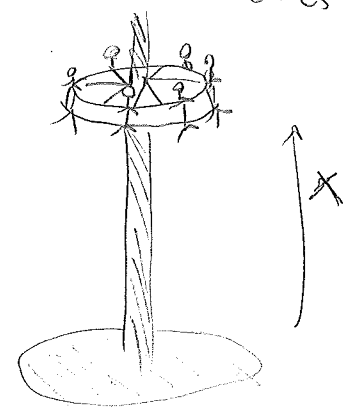
Rate of Change - Introduction to limits W2-W

A common amusement park ride lifts riders to a height then allows them to freefall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 50 metres



Model:

$$f(t) = -5t^2 + 50 \quad \text{for } t \geq 0$$

without intervention, they will reach the ground at $t = \sqrt{10} \approx 3.16$ seconds.

Q: Suppose the designers decide to start decelerating after 2 seconds, how fast will the riders (at the height of 30 metres) be traveling at that time?

I don't like the imperial measuring system!

First attempt at an answer:

Average speed: $v_{avg} = \dfrac{x_2 - x_1}{t_2 - t_1} = \dfrac{\Delta x}{\Delta t}$

Examples:

minus sign say they go down

- $\dfrac{f(2) - f(0)}{2 - 0} = \dfrac{(-5 \cdot 2^2 + 50) - (-5 \cdot 0^2 + 50)}{2 - 0} = -10 \frac{m}{sec}$

But it goes much faster at the second half of the interval.

- $\dfrac{f(2) - f(1)}{2 - 1} = \dfrac{(-5 \cdot 2^2 + \cancel{50}) - (-5 \cdot 1^2 + \cancel{50})}{2 - 1} = -15 \frac{m}{sec}$

- $\dfrac{f(2) - f(1.5)}{2 - 1.5} = \dfrac{(-5 \cdot 2^2 + \cancel{50}) - (-5 \cdot (1.5)^2 + \cancel{50})}{0.5} = -17.5 \frac{m}{sec}$

- $\dfrac{f(2)-f(1.9)}{2-1.9} = \dfrac{(-5\cdot 2^2+50)-(-5\cdot(1.9)^2+50)}{0.1} = -19.5 \frac{m}{sec}$

- $\dfrac{f(2)-f(1.99)}{2-1.99} = -19.95 \frac{m}{sec} = 71.82 \frac{km}{h}$

$\boxed{\text{Desmos demonstration}}$

- $\dfrac{f(2)-f(1.999999)}{2-1.999999} = -19.999995 \frac{m}{sec}$

It looks like the speed at 2 seconds is $20 \frac{m}{sec} \approx 72 \frac{km}{h}$

since $\dfrac{f(2)-f(t)}{2-t}$ gets closer and closer to 20 as t gets closer to 2. This is called instantaneous velocity.

(olympic champion runs at around 40 ft/sec max. speed at luna park drops are around 75 km/h)

$\boxed{\text{Speedometers, Amplifier dial}}$

<u>Informal definition of limit:</u> (Isaac Newton, Gottfried Leibnitz)

Suppose that $f(x)$ is defined for all x near some point $x=a$. If $f(x)$ is arbitrarely close to some (fixed) value L for all x close to (but not including) $x=a$ then we write

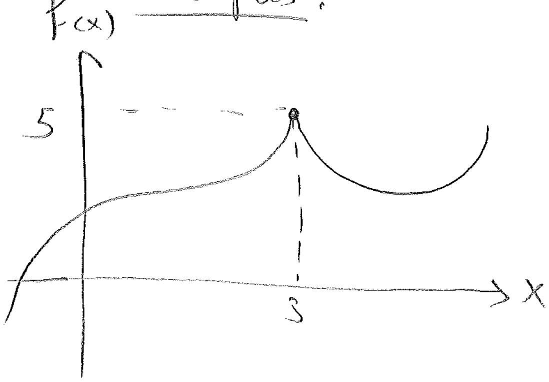
$$\lim_{x\to a} f(x) = L \qquad \left(\text{sometimes } f(x) \xrightarrow[x\to a]{} L\right)$$

"the limit of $f(x)$ as x approaches a is L"

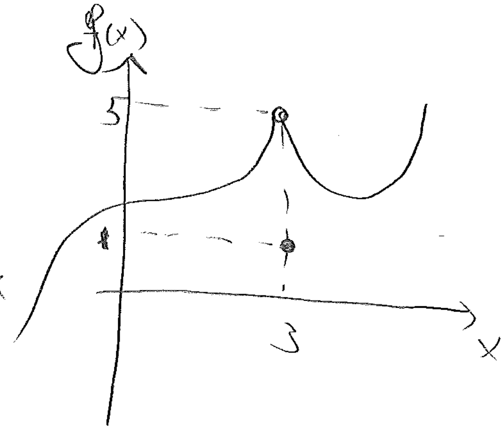
Impertent Remark about this definition:

- The value of $f(a)$ is completely irrelevant to this definition. In fact, f need not be defined at $x=a$ at all.

Examples:

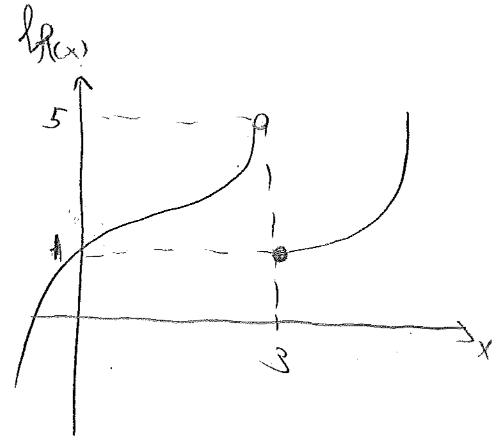


$$\boxed{\lim_{x\to 3} f(x)=5}$$



$$\boxed{\lim_{x\to 3} g(x)=5}$$

even though $f(3)=1\neq 5$



$\lim_{x\to 3} h(x)$ doesn't exist since

f approach 1 from the right and 5 from the left.

The answer wouldn't have changed if $f(3)$ was to be 5 instead.

Anothes famous examples: (Trigonometric functions are not too impertent in our course)

$f(x)=\dfrac{\sin(x)}{x}$ is not defined when $x=0$. $\lim_{x\to 0} f(x)=1$

$g(x)=\sin\left(\frac{1}{x}\right)$ ——— // ——— $\lim_{x\to 0} g(x)$ doesn't exist

$h(x)=x\cdot\sin\left(\frac{1}{x}\right)$ ——— // ——— $\lim_{x\to 0} h(x)=0.$

A few last remarks about the example:

- In our new notations $\lim_{t \to 2} f(t) = -20$

actually we checked what happen only at $t<2$, but if you'll check (!) what happens for $2<t$ you will find the same thing.

- For any function $f(x)$, the average rate of change of f over the interval $[a,b] = \{x \text{ s.t. } a \le x \le b\}$ is defined to be

$$\frac{f(b) - f(a)}{b-a}$$

The instantaneous rate of change of f at $x=a$ is equal to the slope of the tangent line to the graph of f at $x=a$. it is $\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = h}$

The tangent is $\boxed{y = h(x-a) + f(a)}$