Warm-up problems:

- $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = ?$

note that $(\sqrt{1+x}-1)(\sqrt{1+x}+1) = \sqrt{1+x}^2 - 1^2 =$
$(x \geq -1)$

$= 1+x-1 = x$

$\Rightarrow \frac{\sqrt{1+x}-1}{x} = \frac{1}{\sqrt{1+x}+1} \quad (x \geq -1,\ x \neq 0)$

$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$

- $\lim_{x\to 2} \frac{\frac{1}{2x+3} - \frac{1}{7}}{x-2} = \lim_{x\to 2} \frac{\frac{7}{7(2x+3)} - \frac{2x+3}{7(2x+3)}}{x-2} =$

$= \lim_{x\to 2} \frac{\left(\frac{7-(2x+3)}{7(2x+3)}\right)}{x-2} = \lim_{x\to 2} \frac{4-2x}{7(x-2)(2x+3)} = \lim_{x\to 2} \frac{-2}{7(2x+3)} = -\frac{2}{49}$

Continuity:

Q: When can we say $\lim_{x\to c} f(x) = f(c)$?

This situation has a name.

Def: If $\lim_{x\to c} f(x) = f(c)$ we say that f is continuous at c.

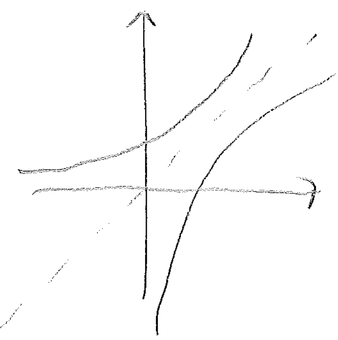
Let I be an interval. (so $I = [a,b], (a,b), [a,b), (a,b], [a,\infty), (a,\infty), (-\infty,b], (-\infty,b), (-\infty,\infty)$)

We say that f is continuous at I if it is continuous at c for all $c \neq a,b$ in I and:

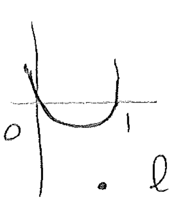
- if $a \in I$ then $\lim_{x\to a^+} f(x) = f(c)$.
- if $b \in I$ then $\lim_{x\to b^-} f(x) = f(c)$.

Examples:

- polynomials: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ is continuous on $(-\infty,\infty)$
- e^x is continuous on $(-\infty,\infty)$



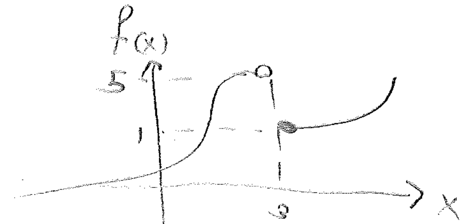
- $\ln(x)$ is continuous on $(0,\infty)$
- $\sqrt[n]{x}$ is continuous on: $[0,\infty)$ if n is even.
  (n integer) $(-\infty,\infty)$ if n is odd.
- $\cos(x)$, $\sin(x)$ continuous on $(-\infty,\infty)$
  $\tan(x)$ on $(-\frac{\pi}{2}+\pi n, \frac{\pi}{2}+\pi n)$ for n integer
- inverse trig. fun. ?
- Every addition, ~~division~~, substraction, mult., div. & comp. of such functions are continuous in "the right domain"



- $\frac{\log(x^2-x)}{x+1}$ is cont. for $x \neq -1$ and $x^2 - x > 0$
  that is $x > 1$ or $x < 0$ with $x \neq -1$
  so on: $(-\infty,-1) \cup (-1,0)$, $(1,\infty)$

One-Sided Limits:

Recall: We saw this example



as an example to the non-existence of a limit.
However, we had a feeling that both 1 and 5 should be a limit of this function. We will remedy this.

Def: Suppose that $f(x)$ is defined for all $x<a$ near $x=a$. If $f(x)$ is arbitrarily close to some value L for all x close to (but not including) $x=a$ with $x<a$ then we write

$$\lim_{x\to a^-} f(x) = L \qquad \text{the \underline{left one-sided limit}}.$$

Similarly, if $f(x)$ is defined for all $x>a$ near $x=a$. If $f(x)$ is arbitrarily close to some value M for all x close to (but not including) $x=a$ with $x>a$ then we write

$$\lim_{x\to a^+} f(x) = M \qquad \text{the \underline{right one-sided limit}}.$$

In the example: $\lim\limits_{x\to 3^-} f(x) = 5$, $\lim\limits_{x\to 3^+} f(x) = 1$

Q: What happens when $\lim\limits_{x\to a} f(x) = L$?

Thm: We have $\lim\limits_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

For $f(x)$ defined for all x near a

Exercise:* Assume $f(x) = \begin{cases} \dfrac{x^3 - A}{x^2 - x}, & x > 1 \\[2ex] \dfrac{x^2 - B^2}{x - 1} + B & x < 1 \end{cases}$

What should be A & B so that $\lim\limits_{x \to 1} f(x)$ exists?

alternatively, what should be A & B so that $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$?

Sol:

$B = \pm 1$ so that $\lim\limits_{x \to 1^-} f(x)$ exists, and then

$\lim\limits_{x \to 1^+} f(x)$ exists only if and only if $A = 1$

and then $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{x^3 - 1}{x^2 - x} = \lim\limits_{x \to 1^+} \dfrac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)} \cdot x} = \lim\limits_{x \to 1^+} \dfrac{x^2 + x + 1}{x} = 3$

on the other hand:

$$\lim_{x \to 1^-} \left(\frac{\overbrace{x^2 - 1}^{(x-1)(x+1)}}{x - 1} + B \right) = \lim_{x \to 1^+} (x + 1 + B) = 2 + B$$

So $2 + B = 3$ implies $B = 1$.

Exercise: For which value of a will $f(x) = \begin{cases} 1 + e^{ax}, & x < 9 \\ \sqrt{x}, & x \geq 9 \end{cases}$

be continuous at 9?

Sol:

$$\lim_{x \to 9^-} f(x) = \lim_{x \to 9^-} (1 + e^{ax}) \overset{\text{cont. of } 1+e^{ax}}{=} 1 + e^{a9}$$

$$\lim_{x \to 9^+} f(x) = \lim_{x \to 9^+} \sqrt{x} \underset{\text{cont. of } \sqrt{x}}{=} \sqrt{9} = 3$$

$f(x)$ will be cont. if $3 = 1 + e^{9a}$

$2 = e^{9a}$

$\ln(2) = 9a$ $\quad \boxed{a = \dfrac{\ln(2)}{9}}$