Review - Continuity

Def: A function $f(x)$ is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$

- — // — from the left at $x=a$ if $\lim_{x \to a^-} f(x) = f(a)$
- — // — right at $x=a$ if $\lim_{x \to a^+} f(x) = f(a)$

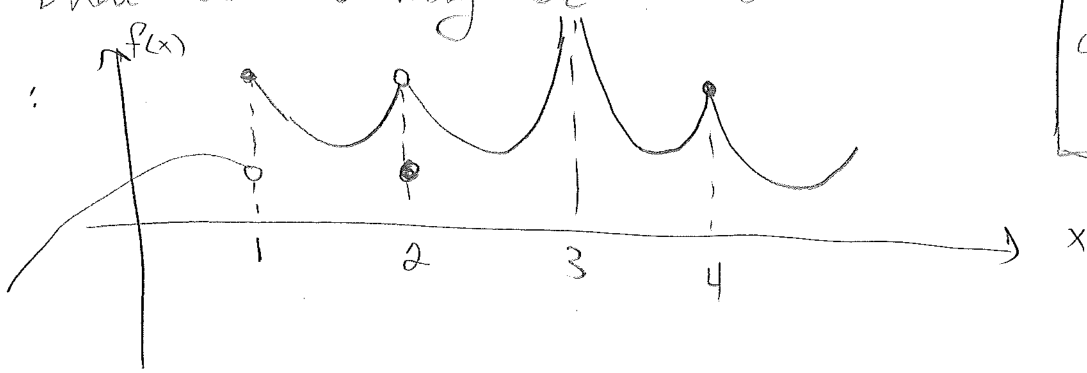
- $f(x)$ is continuous at (a,b) if it is cont. at any $a<x<b$.
- $f(x)$ is continuous at $(a,b]$ if it is cont. at any $a<x<b$ and cont. from the left at $x=b$

HW: Define continuity on $[a,b)$, $[a,b]$

Note that a or b may be $-\infty$ or ∞.

"no gap
no hole
no jump
Can draw without lifting my pen"

Example:



f is continuous at $(-\infty, 1)$, $[1,2)$, $(2,3)$, $(3,\infty)$

- $x=1$ is called a jump discontinuity (no limit but two one sided limits)
- $x=2$ is called a removable discontinuity point (there is a limit but ≠ value or no value)
- $x=3$ is called an essential discontinuity (at least one one-sided limit doesn't exist)

Theorem: assume f,g are continuous at $x=a$ (or an interval I) then so are:

1. sums/differences $f \pm g$
2. Products: $f \cdot g$
3. Quotients. f/g (if $g(a) \neq 0$)
4. Powers f^n (n integers ≥ 0)
5. Roots $\sqrt[n]{f}$ (if n even: $f(a) \geq 0$)

Theorem: (Continuous functions)
The following functions are continuous at their domains: ⓪ Polynomials
① $\sin x$ ② $\cos x$ ③ $\tan x$ ④ $\cot x$ ⑤ $\sec x$ ⑥ $\csc x$
⑦ $\ln(x)$ ⑧ $\sqrt[n]{x}$ ⑨ a^x ⑩ Rational functions

Ex: $f(x) = \dfrac{x^2-3x}{x-1}$

cont on $(-\infty,1)$ & $(1,\infty)$

Ex: For what values of c is

$$f(x) = \begin{cases} cx+1 & x \le 3 \\ cx^2-1, & x>3 \end{cases}$$

Continuous on $(-\infty,\infty)$?

Solution: By theorem: cont on $(-\infty,3]$ & $(3,\infty)$ in order to have cont. on $(-\infty,\infty)$ we need: $\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x)$

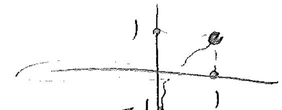
$\lim_{x\to3^-} f(x) = 3c+1$, $\lim_{x\to3^+} f(x) = 9c-1$

$3c+1 = 9c-1$

$6c = 2$

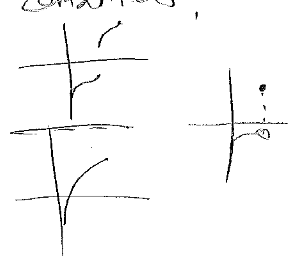
$\boxed{c=\frac{1}{3}}$

Intermediate Value Theorem:

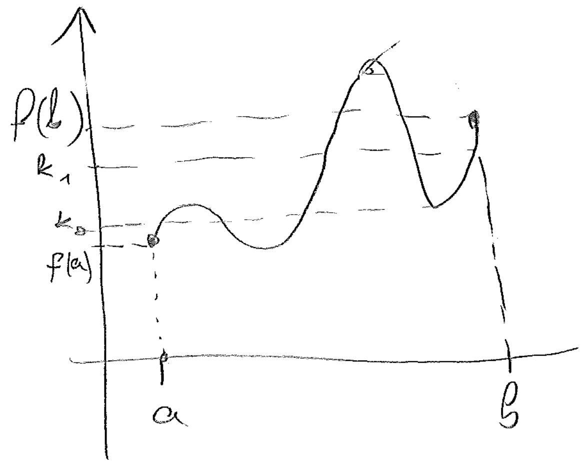


Q: Can you draw a continuous function on $[0,1]$ such that $f(0)=-1$, $f(1)=1$ and f doesn't pass through $y=0$?

Q: What if I lose the conditions?
- Not cont.?
- only cont. on $(0,1)$?
- Take $y=2$ instead?



Theorem: (IVT) Suppose that f is continuous on $[a,b]$. Then for any number k between $f(a)$ and $f(b)$ there exists a number $c \in [a,b]$ so that $f(c)=k$.



(f may have values not between f(a) and f(b) and any value between them is attained <u>at least once</u>

<u>Example:</u> Show that $\sqrt{2}$ exists and must be between 1.4 and 1.5.

<u>Solution:</u> (We need a ~~continuous~~ function that will supply this info.)

<u>Idea:</u> · f continuous on $[1.4, 1.5]$

- $f(1.4) < 0$
- $f(1.5) > 0$
- $f(\sqrt{2}) = 0$
- $f(x) = 0 \Rightarrow x = \sqrt{2}$.

Try: $f(x) = x^2 - 2$.

<u>Indeed:</u>

By IVT there exists $a < c < b$ s.t. $f(c) = 0$ $\Leftarrow$

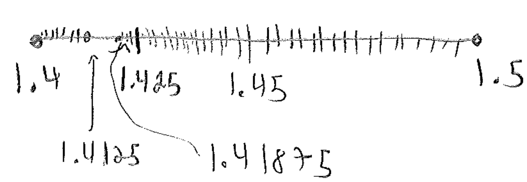
- f is continuous for all x.
- $f(1.4) = 1.4^2 - 2 = \left(\frac{7}{5}\right)^2 - 2 = \frac{49}{25} - 2 = -\frac{1}{25} < 0$
- $f(1.5) = 1.5^2 - 2 = \left(\frac{3}{2}\right)^2 - 2 = \frac{9}{4} - 2 = \frac{1}{4} > 0$

$c = \sqrt{2}$ $\Leftarrow$

- $f(\sqrt{2}) = \sqrt{2}^2 - 2 = 2 - 2 = 0$
- $f(x) = 0 \Leftrightarrow x^2 = 2$ If $x \geq 0$ then $x = \sqrt{2}$ (o.w. $x = -\sqrt{2}$)

and hence $1.4 < \sqrt{2} < 1.5$.

<u>Bisection method:</u>



$\sqrt{2} \approx 1.4142135\ldots$

x	f(x)
1.4	$-1/25$
1.5	$1/4$
1.45	0.1025
1.425	0.030625
1.4125	$-0.0048437$5
1.41875	0.1285156…