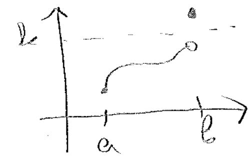
Some more on IVT:

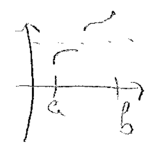
Theorem: Let $f(x)$ be a cont. function on $[a,b]$. For any k between $f(a)$ and $f(b)$ there exists $a \leq c \leq b$ such that

$$f(c) = f(k).$$



What can go wrong? (When can't we apply the theorem)

- $f(x)$ cont. only on (a,b)



- $f(x)$ not continuous...
- $k > f(b)$ or $k < f(a)$

Example of usage: Prove that the equation $e^x - \sin(x) = 0$ has a solution $x = c$.

Proof: Let $f(x) = e^x - \sin(x)$, Note that:

(⓪ $f(x)$ is continuous on $(-\infty, \infty)$)

① $f(-\frac{\pi}{2}) = e^{-\frac{\pi}{2}} - 1 < 0$

② $f(0) = e > 0$

③ $f(x)$ is continuous on $[-\frac{\pi}{2}, 0]$

Hence, by IVT, there exists $x = c$, $-\frac{\pi}{2} < c < 0$ s.t. $f(c) = 0$.

Derivatives:

Recall: Given $f(x)$ defined near $x=a$, if the limit

$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists we denote it by $f'(a)$ and

call it the derivative of $f(x)$ at $x=a$.

Another notation: $\frac{d}{dx}(f)(a)$ Fact: $f(x)$ is cont. at $x=a$.

- Example: Let $f(x)=x^n$ for a positive integer n.

What is $f'(a)$?

Solution: $\frac{f(x)-f(a)}{x-a} = \frac{x^n-a^n}{x-a}$ do we know this expression?

Recall the arithmetic sum: $1+q+q^2+q^3+q^4+\dots+q^{n-1} = \frac{q^n-1}{q-1}$

So: $\frac{x^n-a^n}{x-a} = \frac{1-\left(\frac{a}{x}\right)^n}{1-\frac{a}{x}} \cdot \frac{x^n}{x} = \left(1+\frac{a}{x}+\left(\frac{a}{x}\right)^2+\dots+\left(\frac{a}{x}\right)^{n-1}\right)\cdot x^{n-1}$

can assume $x\neq 0$

$\xrightarrow[x\to a]{} \underbrace{(1+1+1+\dots+1)}_{n \text{ times}} \cdot a^{n-1} = na^{n-1}$

(Remark: This proof is harder for the other defn. of derivative)

Remark: If $f(x)$ has a derivative on an interval I (we say f is differentiable on I) then a function $f'(x)$ is defined. So we may write $\boxed{f'(x)=nx^{n-1}}$

Theorem: Let f,g be differentiable functions at $x=a$ then:

(0) If $h(x)$ is constant near $x=a$ then $h'(a)=0$.

(1) $(f\pm g)'(a) = f'(a)\pm g'(a)$ ← Explain

proof: (0) $\lim_{x\to a}\frac{h(x)-h(a)}{x-a} = \lim_{x\to a}\frac{0}{x-a} = \lim_{x\to a} 0 = 0$.

(1) $\lim_{x\to a}\frac{(f\pm g)(x)-(f\pm g)(a)}{x-a} = \lim_{x\to a}\frac{f(x)\pm g(x)-f(a)\mp g(a)}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \pm \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a)\pm g'(a)$

② For a constant c let $g(x) = c f(x)$, then

$$g'(a) = (cf)'(x) = c \cdot f'(x).$$

HW: proof this.

Who knows what this means

Corollary: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$

Then $f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$.

HW: prove this using the theorem and previous example.

Theorem: For $a > 0$ let $f(x) = a^x$. Then $f'(x) = a^x \cdot \ln(a)$.

W4-L1

Exercise: Find a so that $f'(x) = f(x)$ for any x.

Solution: We want $a^x = a^x \cdot \ln(a)$ for any x.

since $a^x \neq 0$ it's equiv. to $\ln(a) = 1$, i.e. $\boxed{a = e}$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

[In fact, this is the only differentiable function satisfying this condition and $f(0) = 1$.]

Exercise: Compute $\frac{d}{dx}(e^{3x+1})$.

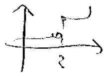
Solution: $e^{3x+1} = e \cdot e^{3x} = e \cdot (e^3)^x$

Hence: $\frac{d}{dx}(e^{3x+1}) = e \cdot \frac{d}{dx}(\underbrace{(e^3)}_{a}{}^x) = \underbrace{e \cdot (e^3)^x}_{e^{3x+1}} \cdot \underbrace{\ln(e^3)}_{3} = 3e^{3x+1}$.

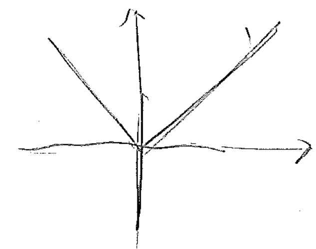
Remark: Note that: $\frac{d(3x+1)}{dx} = 3$. We'll come back to it later this week.

③

Intermezzo: How can a (can't) function fail to be differentiable?

Example!  Not diff at $x=3$.

Example: Let $f(x) = |x| = \begin{cases} x, & x>0 \\ 0, & x=0 \\ -x, & x<0 \end{cases}$



Since a derivative is a local property, $f(x)$ is differentiable at any $x \neq 0$. What happens for $x=0$?

Does $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists?

Let's look at one sided limits:

- $\lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \frac{x-0}{x-0} = \lim\limits_{x\to 0^+} 1 = 1.$
- $\lim\limits_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \frac{-x-0}{x-0} = \lim\limits_{x\to 0^-} (-1) = -1.$ $\neq$

So $f(x)$ is cont. but not diff. at $x=0$.

What is the "cause" of non-diff. here? Roughly speaking, this is because f has an "edge". Remember that the derivative is the slope of the tangent, here there is no reasonable candidate for a tangent.

Example: Weierstrass' function.

Trigonometric functions:

Theorem: ① $\frac{d}{dx}(\sin x) = \cos x$ ② $\frac{d}{dx}(\cos x) = -\sin x$ ③ $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2(x)}$

④ $\frac{d}{dx}(\cot x) = -\frac{1}{\sin(x)^2}$ HW: sec... cosec...

Proof: ① $\frac{d}{dx}(\sin x) = \lim\limits_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim\limits_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \overset{\cos x}{\overset{\shortparallel}{\lim\limits_{h\to 0}\cos\left(\frac{2x+h}{2}\right)}} \cdot \overset{1}{\overset{\shortparallel}{\lim\limits_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{h/2}}}$

② the same.

③ $\frac{d}{dx}(\tan x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos(x)\cdot\cos(x) - (-\sin(x))\cdot\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2(x)}$

Product and Quotients Rules:

Last time we've discussed the derivative of sums, differences, scalar mult,

Theorem: Let $f(x)$ and $g(x)$ be differential functions at $x=a$

① $(f\cdot g)'(a) = f'(a)g(a) + f(a)\cdot g'(a)$ (product rule / Leibnitz rule)

② If $g(a) \neq 0$ then $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}$

HW: Write a proof.

Examples: ① Let $f(x) = (x^2+3x+1)(2x^2-3x+1)$

$$f'(x) = (2x+3)(2x^2-3x+1) + (x^2+3x+1)(4x-3) = (4+4)x^3 + (-6+6-3+12)x^2 + (2-9-9+4)x + 0$$

other way: $f(x) = 2x^4 + (-3+6)x^3 + (2-9+1)x^2 + (-3+3)x + 1 = 2x^4+3x^3-6x^2+1$

$$f'(x) = 8x^3 + 9x^2 - 12x$$

If there's time

② $f(x) = x\ln(x)$ Fact: $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

$f'(x) = \ln(x) + x\cdot\frac{1}{x} = \ln(x)+1$ Q: Can you suggest $f(x)$ s.t. $f'(x) = \ln(x)$?

③ $f(x) = x\ln x - x$

$f'(x) = \ln(x) + x\cdot\frac{1}{x} - 1 = \ln(x)$

④ $f(x) = \frac{5x^2}{x^2+1}$

$$f'(x) = \frac{10x(x^2+1) - (2x)\cdot(5x^2)}{(x^2+1)^2} = \frac{\cancel{10x^3} + 10x - \cancel{10x^3}}{(x^2+1)^2} = \frac{10x}{(x^2+1)^2}.$$

⑤ $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\cdot\cos x - (-\sin x)\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

Anti-Example: $f(x) = x\cdot|x|$. $|x|$ is not diff. at $x=0$ but the product still can be.

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x<0 \end{cases}$$

$$\begin{cases} \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x = 0 \\ \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} (-x) = 0 \end{cases}$$

So $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0.$

Warm-Up:

① Find a point $x=a$ such that the slope of tangent of $f(x) = e^{3x} + e^{-2x}$ at $x=a$ is 0.

Solution: We've seen that $f'(x) = \frac{d}{dx}\left((e^3)^x\right) + \frac{d}{dx}\left((e^{-2})^x\right) = 3\cdot e^{3x} - 2e^{-2x}$.

The slope of the tangent of $f(x)$ at $x=a$ is $f'(a)$.
We solve $f'(a)=0$ for a.

$$3e^{3a} - 2e^{-2a} = 0$$
$$3e^{3a} = 2e^{-2a}$$
$$e^{5a} = \frac{2}{3}$$
$$\boxed{a = \frac{1}{5}\ln\left(\frac{2}{3}\right)}$$

② Find a value a such that $f(x) = \begin{cases} x^2+x+1, & x \le 0 \\ e^{ax}, & x > 0 \end{cases}$ is differentiable at $x=0$.

Solution: $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists if and only if the two one-sided limits exists.

- $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{e^{ax}-1}{x-0} = \frac{d}{dx}\left(e^{ax}\right)\Big|_{x=0} = a$
- $\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{(x^2+x+1)-(0+0+1)}{x-0} = \frac{d}{dx}\left(x^2+x+1\right)\Big|_{x=0} = (2\cdot x+1)\Big|_{x=0} = 1$

So $f(x)$ is diff. at $x=0$ iff. $a=1$.

Chain rule:

Q: $\frac{d}{dx}(e^{x^2}) = ?$ (We are riding along e^x slower)

Theorem: Let $f(x)$ be diff. $x=b$ and $g(x)$ be diff. at $x=a$ with $g(a)=b$. Then

$$(f\circ g)'(a) = f'(g(a))\cdot g'(a) \quad , \text{ i.e.}$$

$$F'(a) = f'(b)\,g'(a) \quad \text{for} \quad F(x) = f(g(x)).$$

Solution: $\frac{d}{dx}(e^{x^2}) = f'(g(x))\cdot g'(x)$ $\qquad f(x)=e^x \quad f'(x)=e^x$, $g(x)=x^2 \quad g'(x)=2x$

$$= e^{x^2}\cdot 2x.$$

Example: $e^{\ln(x)} = x$ for any $x>0$ $\qquad f(x)=e^x \quad f'(x)=e^x$, $g(x)=\ln(x) \quad g'(x)=?$

So $\frac{d}{dx}(\ln(e^x)) = \frac{d}{dx}(x)$

$$\underbrace{f'(g(x))}_{e^{\ln(x)}}\cdot g'(x) = 1$$

So $x\cdot g'(x)=1$ ~~Hence~~ Hence $\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$

Example: $(\sqrt[n]{x})^n = x$ for any x (>0 if n is even)

$f(x)=x^n$, $g(x)=\sqrt[n]{x}$

$f'(x)=nx^{n-1}$, $g'(x)=?$

Inverse function: If $f(g(x))=x$ for x near $x=a$, $b=g(a)$ and $f'(b)\neq 0$ then: $\boxed{g'(a) = \frac{1}{f'(b)}}$

Der. of

$$f'(g(x))\,g'(x) = 1$$

$$n\underbrace{(\sqrt[n]{x})^{n-1}}_{x^{\frac{n-1}{n}} = x^{1-\frac{1}{n}}}\cdot g'(x) = 1$$

Hence: $\boxed{\frac{d}{dx}(\sqrt[n]{x}) = \frac{1}{n}\cdot x^{\frac{1}{n}-1}}$

Theorem: For any real number r

$$\boxed{\frac{d}{dx}(x^r) = rx^{r-1}}$$

Other examples:

- $$\frac{d}{dx}\left((x^2+1)^{10}\right) = 10(x^2+1)^9 \cdot (2x+1)$$

- $$\frac{d}{dx}\left(\ln\left(\frac{x}{x+1}\right)\right) = \frac{1}{\left(\frac{x}{x+1}\right)} \cdot \frac{(x+1)-x}{(x+1)^2} = \frac{x+1}{x} \cdot \frac{1}{(x+1)^2} = \frac{1}{x(x+1)}$$

Ex: Find $f(x)$ such that $f'(x) = \ln(x)$.

- $$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(x\ln(x)\right) = \ln(x) + x\cdot\frac{1}{x} = \ln(x) + 1$$

$$\boxed{\frac{d}{dx}\left(x\ln(x) - x\right) = \ln(x) + 1 - 1 = \ln(x).}$$

$\boxed{f(x) = x\ln(x) - x} + c$ Called "inverse derivative"