Trigonometric & Inverse Trig. Functions

Derivative table on page 101

Theorem:

① $\dfrac{d \sin x}{dx} = \cos x$ ③ $\dfrac{d \tan x}{dx} = \dfrac{1}{\cos^2 x}$

② $\dfrac{d \cos x}{dx} = -\sin x$ ④ $\dfrac{d \cot x}{dx} = -\dfrac{1}{\sin^2 x}$

Partial proof: ①

$$\lim_{x \to a} \frac{\sin(x) - \sin(a)}{x-a} = \lim_{x \to a} \frac{2\sin\left(\frac{x-a}{2}\right)\sin\left(\frac{x+a}{2}\right)}{x-a}$$

$$= \underbrace{\lim_{x \to a} \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}}_{1} \cdot \underbrace{\lim_{x \to a} \sin\left(\frac{x+a}{2}\right)}_{\sin(a)} = \sin(a)$$

③ $(\tan x)' = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{(\cos x)^2 - (-\sin x)\sin x}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$.

$\arcsin(x)$, $\arccos(x)$ are defined for $-1 \le x \le 1$

Theorem:

① $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$ ③ $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$

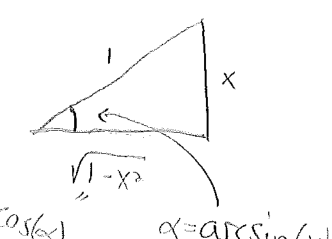
② $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$ ④ $\dfrac{d}{dx}(\operatorname{arccot} x) = -\dfrac{1}{1+x^2}$

proof:

$\sin(\arcsin x) = x$

$\cos(\arcsin x)\,(\arcsin x)' = 1$

$(\arcsin x)' = \dfrac{1}{\cos(\arcsin x)} = \dfrac{1}{\sqrt{1-x^2}}$

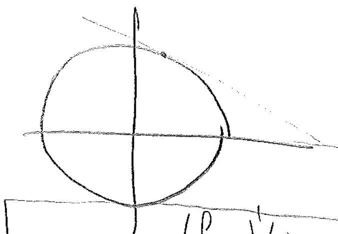


Example: $(\sin(\cos(e^{x^2})))' =$

$= \cos(\cos(e^{x^2})) \cdot [\cos(e^{x^2})]' = \cos(\cos(e^{x^2}))(-\sin(e^{x^2})) \cdot 2x e^{x^2}$.

Implicit Diferentiation

Compute the tangent to the circle $x^2+y^2=1$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



Solution 1: $y = \sqrt{1-x^2}$

$$y'(x) = \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}\cdot(-2x)$$

$$\frac{dy}{dx}\left(\frac{1}{2}\right) = \frac{1}{2\frac{\sqrt{3}}{2}}(-1) = -\frac{1}{\sqrt{3}}$$

$$l(x) = -\frac{1}{\sqrt{3}}\left(x-\frac{1}{2}\right)+\frac{\sqrt{3}}{2}$$

$\boxed{(f\circ g)'(x) = f'(g(x))\cdot g'(x)}$

Remark: Alternative notation to the chain rule:

$$\frac{d(f\circ g)}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$$

Leibnitz notation

Solution 2: $x^2+y^2=1$

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$

$$\cancel{2}x + \cancel{2}y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{So} \quad \frac{dy}{dx}\left(\frac{1}{2}\right) = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}.$$

Q: What is the line eq. of the radius?

Answer: Let $l(x) = Ax+B$ be a line s.t. $A \neq 0$. Then the normal (perpendicular) line has a slope $-\frac{1}{A}$

So $R(x) = \sqrt{3}\left(x-\frac{1}{2}\right)+\frac{\sqrt{3}}{2} = \sqrt{3}x.$

Example: • Find y' give $y^3 + \sin(y) = 6 - x^3$

$y'(3y^2 + e^y) = 3y^2 y' + \cos(y) \cdot y' = -3x^2$

$$y' = \frac{-3x^2}{3y^2 + \cos(y)}$$

• What is the tangent at $(\sqrt[3]{6}, 0)$

$l(x) = -3(\sqrt[3]{6})^2 (x - \sqrt[3]{6})$

$$y'(\sqrt[3]{6}) = \frac{-3(\sqrt[3]{6})^2}{3 \cdot 0 + \cos(0)} = -3(\sqrt[3]{6})^2$$

Example: Find the derivative of $x^{\frac{m}{n}}$ m, n integers, $n \neq 0$.

$y = x^{\frac{m}{n}} \rightsquigarrow y^n = x^m$

$n y^{n-1} y' = \frac{d}{dx}(y^n) = \frac{d}{dx}(x^m) = m x^{m-1}$

So $y' = \frac{m x^{m-1}}{n y^{n-1}} = \frac{m}{n} \frac{x^{m-1}}{y^n} \cdot y =$

$$= \frac{m}{n} \frac{x^{m-1} \cdot x^{\frac{m}{n}}}{x^m} = \frac{m}{n} x^{\frac{m}{n} - 1}$$

Example: Find y' given $\sin(x^2y^2) + y^3 = x + y$

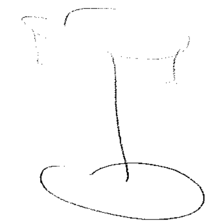
Solution: $\frac{d}{dx}\left(\sin(x^2y^2) + y^3\right) = \cos(x^2y^2)\frac{d}{dx}(x^2y^2) + 3y^2y'$

$= \cos(x^2y^2)\left[2xy^2 + x^2 2yy'\right] + 3y^2y'$

$\frac{d}{dx}(x+y) = 1 + y'$

Solving for y' $\Rightarrow$ $y' = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2) + 3y^2 - 1}$

Higher Derivatives:



$f(t) = -5t^2 + 50$

$f'(t)$, velocity (inst.) $= -10 \cdot t \ \left(\frac{m}{sec}\right)$

$f''(t) = -5 \ \left(\frac{m}{sec^2}\right)$ is ~~the~~ called acceleration.

$(f')'(t)$

Def: Let $f(x)$ be a diff. function on I. Then:

- The second derivative of $f(x)$ is: $f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2}$
  (Here $f'(x)$ is diff. on I two)
- The third der. of $f(x)$ is $f'''(x) = \frac{d}{dx}(f''(x)) = \frac{d^3 f(x)}{dx^3}$
- ⋮ The n^{th} —— // —— $f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d^n f(x)}{dx^n}$.

Examples: ① $f(x) = 4x^2$, $f'(x) = 8x$, $f''(x) = 8$, $f'''(x) = 0$.

② $f(x) = 5e^x$, $f'(x) = 5e^x$, $f''(x) = 5e^x$, ... $f^{(n)}(x) = e^x$

③ $f(x) = \cos(x)$ $f^{(2n)} = (-1)^n \cos(x)$, $f^{(2n-1)}(x) = (-1)^n \cdot \sin(x)$.

What does the second derivative say? It will help us (in the future) to determine Concavity (& points of inflection) and the nature of local extremum points.