- Quiz 2 in the MLC, on Thursday bring UBC card and remember to ask for: MATH 104 SECTION 108 QUIZ #2.
- Common mistakes on section website
- Tomorrow Midterm $18^{45}-19^{45}$ Buch. A203.

(UBC card, go through policy on website)

Warm Up:

① Compute y' for $y=x^x$

② Compute y' for $y=\frac{x^2\sqrt{1+x}}{(2+\sin(x))^7}$

Solutions: ① $y=x^x=(e^{\ln x})^x=e^{x\cdot\ln(x)}$

$$y'=\underbrace{e^{x\ln(x)}}_{x^x}\cdot(x\ln(x))'=x^x\left(\ln(x)+x\cdot\tfrac{1}{x}\right)=x^x(\ln(x)+1)$$

② can do: y' = awful quotient + chain rules

Let's do something else.

$$\ln(y)=\ln\left(\frac{x^2\sqrt{1+x}}{(2+\sin(x))^7}\right)$$

$$\frac{1}{y}\cdot y'=\frac{d}{dx}\ln(y)=\frac{d}{dx}\left[\ln\left(\frac{x^2\sqrt{1+x}}{(2+\sin(x))^7}\right)\right]=$$

$$\boxed{y'=\frac{\left(\frac{2}{x}+\frac{1}{2(1+x)}-\frac{7\cos(x)}{2+\sin(x)}\right)}{\left(\frac{x^2\sqrt{1+x}}{(2+\sin(x))^7}\right)}}$$

$$=\frac{d}{dx}\left[\ln(x^2\sqrt{1+x})-\ln((2+\sin(x))^7)\right]$$

$$=\frac{d}{dx}\left[2\ln(x)+\frac{1}{2}\ln(1+x)-7\ln(2+\sin(x))\right]=$$

$$=\frac{2}{x}+\frac{1}{2(1+x)}-7\frac{\cos(x)}{2+\sin(x)}$$

Related Rates:

Q1: Suppese that air is being pumped inte a spherical baloon at rate 100 cm^3/sec. How fast is the diameter increasing when it is 50 cm?

Solution:

(assume spherical baloon)

Volume: $V(t)$ $\quad V'(t_0) = 100 \ cm^3/sec$

$$V(t) = \frac{4\pi r(t)^3}{3} = \frac{\pi d(t)^3}{6} \leftarrow \text{diameter}$$

By the chain rule: $V'(t) = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi d(t)^3}{6}\right) = \frac{\pi}{6}\cdot\left(3d(t)^2 \cdot d'(t)\right)$

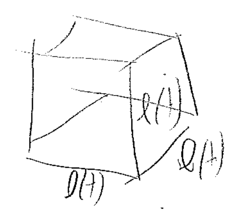
So: $d'(t_0) = \frac{6V'(t_0)}{3\pi d(t_0)^2} = \frac{6\cdot 100}{3\pi\cdot 50^2} = \frac{2}{25\pi} \ \frac{cm}{sec}$

Q2: An ice cube that is 3 cm on each side is melting at the rate of 2 ~~cubic cm/min~~ cm^3/min. How fast is the length of each side decreasing?

Sol:

(assume it stays cubic)

$$V(t) = l(t)^3$$

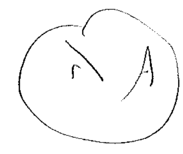


$$V'(t) = 3\, l(t)^2 \cdot l'(t)$$

$$l'(t_0) = \frac{V'(t_0)}{3\, l(t_0)^2} = \frac{2}{3\cdot 3^2} \ \frac{cm}{min}$$

Q3: A camister of oil is forming circular oil slick around itself in the ocean in such a way that the area is increasing at a rate of 2 m^2/min. How fast is the radius of the slick increasing when its radius is 10 meters?

Sol: Area: $A(t) = \pi r(t)^2$ $\quad A'(t_0) = 2 \quad r(t_0) = 10$



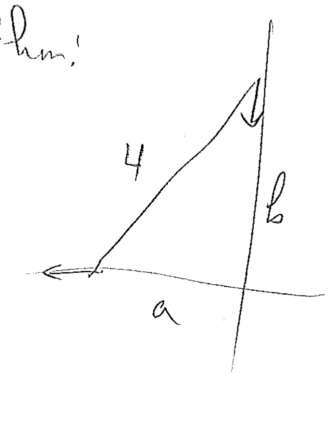
$$A'(t) = 2\pi r(t) \cdot r'(t)$$

$$r'(t_0) = \frac{A'(t_0)}{2\pi r(t_0)} = \frac{2}{2\pi\, 10} = \frac{1}{10\pi}.$$

Q4: One end of a 4 meter is on the ground and the other end rests on a vertical wall. If the bottom end slides from the wall at the rate of 1 m/sec how fast is the top of the ladder ~~traveling~~ sliding down the wall when the bottom of the ladder is 2 meters from the walls

Solution: $a(t)^2 + b(t)^2 = 4^2$ $\quad$ pythagoras theorem!

Deriving: $2a a' + 2b \cdot b' = 0$ $\quad a'(t) = 1$, $a(t_0) = 2$, $b(t_0) = \sqrt{16-4} = \sqrt{12}$



So $b' = \frac{-2 \cdot a \cdot a'}{2b} = -\frac{a}{b} \cdot a'$

$$b'(t_0) = -\frac{2}{\sqrt{12}} \cdot 1 = -\frac{1}{\sqrt{3}} \frac{m}{sec}.$$

why minus sign?

Warm-Up: $y = \sin(x)^{\cos(x)} =$

$y' = ?$

Sol: $y = e^{\cos(x)\ln(\sin(x))}$

$$y' = e^{\cos(x)\ln(\sin(x))} \cdot \left[\cos(x)\ln(\sin(x))\right]' =$$

$$\frac{1-\sin^2(x)}{\sin(x)}$$

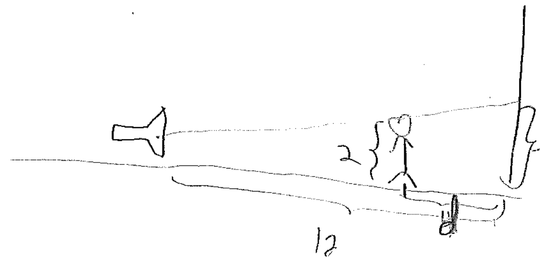
$$= \sin(x)^{\cos(x)} \left(-\sin(x)\ln(\sin(x)) + \cos(x) \cdot \frac{\cos(x)}{\sin(x)}\right).$$

$$= \sin(x)^{\cos(x)}\left(-\sin(x)\ln(\sin(x)) + \frac{1}{\sin(x)} - \sin(x)\right)$$

Q5: A spotlight on the ground shines on a wall 12 meters away. If a two meters man walks from the spotlight to the wall at a speed of 1.6 m/sec, how fast is the length of his shadow on the wall decreasin when he is 4 meters from the wall?

Sol:

$$\frac{h(t)}{2} = \frac{12}{d(t)}$$ Tales' theorem



$$\frac{h'}{2} = -\frac{12}{d^2}\cdot d'$$

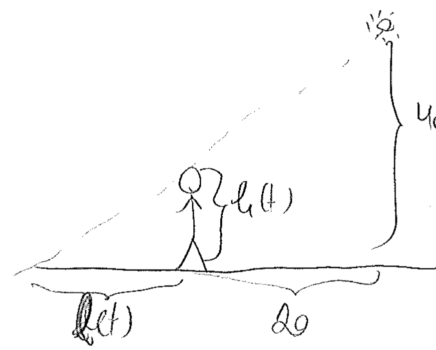
$h(t_0) = ?$

$d'(t_0) = 1.6$

$d(t_0) = 4$

$$h'(t_0) = -\frac{24}{4^2}\cdot 1.6 = -2.4 \text{ m/sec.}$$

Q6: While in wonderland, Alice eats a cookie that makes her grow taller at a rate of ½ m/sec. If she is standing 20 meters from a light which is 10 meters tall how fast is the length of her shadow changing when she is 5 meter tall?



$$\frac{l(t)}{h(t)} = \frac{20 + l(t)}{40}$$

$h'(t_0) = \frac{1}{2}$

$h(t_0) = 5$

$l'(t_0) = ?$

$$40\, l(t) = (20 + l(t))\, h(t)$$

$$40\, l' = 20h' + l'h + l\cdot h'$$

$$l'(t_0) = \frac{20\, h'(t_0) - l(t_0)\, h'(t_0)}{40 - h(t_0)}$$

$$= \frac{20\cdot\frac{1}{2} - \frac{20}{7}\cdot\frac{1}{2}}{40 - 5} =$$

$$\frac{l(t_0)}{5} = \frac{20 + l(t_0)}{40}$$

$$8\, l(t_0) = 20 + l(t_0)$$

$$l(t_0) = \frac{20}{7}$$

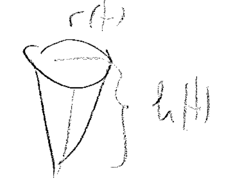
Q7: There is a cone-shaped icicle on your roof and you want to know how fast it is melting. You found that the icicle's volume is decreasing at the rate of 10 cm^3/hour and its radius is decreasing at the rate of 0.4 cm/hour. If the icicle is currently 5cm in radius and 14 cm long, how fast is its height changing?

Sol: $V(t) = \frac{1}{3}\pi\, r(t)^2 \cdot h(t)$

$V'(t_0) = -10$

$r'(t) = -0.4$

$h(t_0) = 14,\ r(t_0) = 5$



$$V' = \frac{1}{3}\pi\left(2r \cdot r' h + r^2 h'\right)$$

$$h'(t_0) = \frac{\frac{3}{\pi}V'(t_0) - 2 \cdot r(t_0) \cdot r'(t_0)\, h(t_0)}{r(t_0)^2} =$$

$$= \frac{-\frac{3}{\pi}10 + 2 \cdot 5 \cdot 0.4 \cdot 14}{5^2} = \frac{56\pi - 30}{25\pi}$$

Q8: General Farms Cereals makes q thousand packs of Fruit Loops Cereal in the marketplace each week when the wholesale price is \$p per box. The relationship between q and p is governed by the supply/demand eq. $6q^2 - 5qp + 2p^3 = 5$

How fast is the supply of cereals changing when the price per box is \$6.50, the quantity sup. is 10,000 boxes ($q=10$) and the whole sale price per box is increasing at the rate of 10¢ per box each week?

Sol: $p(t_0) = 6.5$

$q(t_0) = 10$

$p'(t_0) = 0.1$ \$/week

$q'(t_0) = ?$

$$12q \cdot q' - 5q'p - 5qp' + 6p^2 \cdot p' = 0$$

$$q'(t_0) = \frac{-6p^2(t_0)p'(t_0) + 5q(t_0)p'(t_0)}{12q(t_0) - 5p(t_0)}$$

round numbers → (decrease of 233 boxes/week) $= \frac{6 \cdot 6.5^2 \cdot 0.1 + 5 \cdot 10 \cdot 0.1}{12 \cdot 10 - 5 \cdot 6.5} = -0.23257$

Q9: You observe a speed boat that travels ~~at a speed of 100 km/h~~ on a strait line. What is the angular velocity of your head when the boat is closest to you if it is 500 meters from you and its linear velocity is 100 km/h?

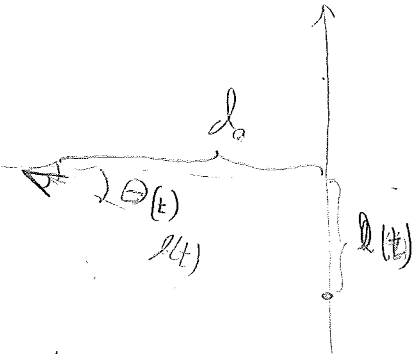
Solution 1:

$\ell(t)$ $\qquad d_o = \frac{1}{2}\,km$

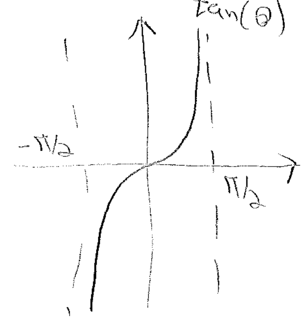
$\ell'(t_o) = 100\ km/h$

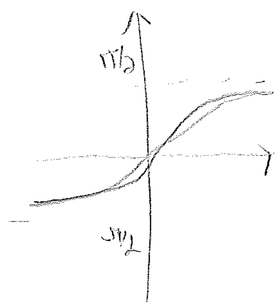
$\ell(t_o) = \ell_o$

$$\tan(\theta(t)) = \frac{\ell(t)}{d_o}$$



$$\theta(t) = \arctan\left(\frac{\ell(t)}{d_o}\right);$$





$$\theta'(t) = \frac{1}{1+\left(\frac{\ell(t)}{d_o}\right)^2}\left(\frac{\ell(t)}{d_o}\right)'$$

$$= \frac{\ell'(t)/d_o}{1+\left(\frac{\ell(t)}{d_o}\right)^2} =$$

$$\ell(t)^2 + d_o^2 = \ell(t)^2$$

$$\left(\frac{\ell(t)}{d_o}\right)^2 = \frac{\ell(t)^2 - d_o^2}{d_o^2} = \left(\frac{\ell(t)}{d_o}\right)^2 - 1$$

$$= \frac{\ell'(t)/d_o}{1+\left(\frac{\ell(t)}{d_o}\right)^2 - 1} = \frac{\ell'(t)/d_o}{\left(\ell(t)/d_o\right)^2}$$

$$\theta'(t_o) = \frac{\ell'(t_o)/d_o}{(d_o/d_o)^2} = \ell'(t_o)/d_o = 100/\tfrac{1}{2} = \underline{\underline{200\ rad/h}}.$$

$\theta(t_o) = 0$

$\cos(\theta(t_o)) = 1$

Solution 2: $\tan(\theta(t)) = \frac{\ell(t)}{d_o} \rightsquigarrow \frac{1}{\cos(\theta(t))^2}\cdot\theta'(t) = \frac{\ell'(t)}{d_o}$

$$\rightsquigarrow \theta'(t_o) = \frac{\ell'(t_o)}{d_o}$$

Remark on Notations: $\arctan(x) = \tan^{-1}(x)$

$$\cot(x) = \tan(x)^{-1}.$$