Average Cost & Marginal Cost:

Example: A factory produces chairs, the fixed cost of manufacture is \$10,000 and the cost of making a chair is \$5.

The cost function is $C(q) = 10{,}000 + 5q$.

The average cost of making a chair is $\frac{C(q)}{q} = \frac{10{,}000 + 5q}{q}$

This is the effective cost of each chair $= \frac{10{,}000}{q} + 5$

So the more chairs you sell, then the fixed cost will be lower pair chair.

$$C_{avg}(q) = \frac{C(q)}{q}$$

Similarly: $R_{avg}(q) = \frac{R(q)}{q}$ ← Average Revenue

$P_{avg}(q) = \frac{P(q)}{q}$ ← Average profit.

Q: If you make q chairs, what is the price of making 1 more chair?

A: In this example, the answer is \$5 always

But here $C(q)$ is a linear function so it's easy.

Q: what if $C(q)$ is not linear?

A: $C_{1\,more}(q) = C(q+1) - C(q)$

But it's not always easy to compute that.
We should have other ways to approximate $C_{1\,more}(q)$

In the example:

$$5 = C(q+1) - C(q) \quad \text{but also} \quad 5 = \text{slope} = \frac{dC}{dq}.$$

We define: $M.C.(q) = \frac{dC}{dq}$ Marginal Cost.

For a linear Cost function $M.C.(q) = C_{1\,more}(q)$

Example: $C(q) = 10{,}000 + 5\sqrt{q}$

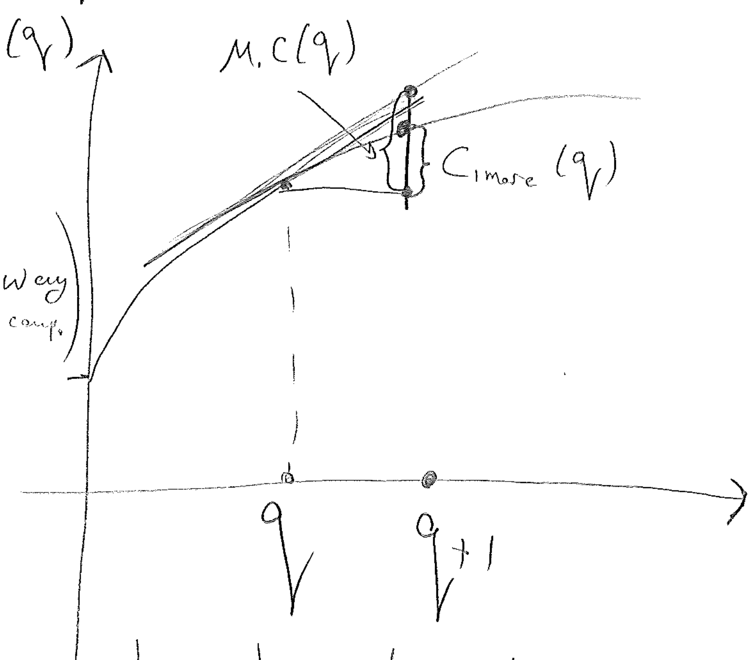
Q: How bad of an approx. is $M.C.(q)$?

$$M.C.(q) = \frac{5}{2\sqrt{q}}$$

(taking one sq. root is way faster in a comp.)

$$C_{1\,more}(q) = 5(\sqrt{q+1} - \sqrt{q})$$

$$C_{avg}(q) = \frac{10{,}000}{q} + \frac{5}{\sqrt{q}}.$$



q	$C_{1\,more}(q)$	$M.C.(q)$	$C_{avg}(q)$
10	0.77...	0.79...	1,001.58...
100	0.249..	0.25..	100.5
1,000	0.079...	0.079...	10.158...
10,000	0.0249...	0.025	1.05

~~100,000~~	~~0.0079..~~	~~0.0079..~~	~~0.1158..~~
~~1,000,000~~	~~0.00249..~~	~~0.0025~~	~~0.015~~
~~1000,000,000~~	~~0.00025~~	~~0.00025~~	~~0.0006~~

So, $M.C(q)$ is a good approx. of $C_{1\,more}$ especially as q grows.

Intuitive explanation:

$$C_{1\,more} = C(q+1) - C(q) = \frac{C(q+1)-C(q)}{1} =$$

$$= \frac{C(q+1)-C(q)}{q+1-q} \underset{\substack{\text{when}\\ q \text{ is big}}}{\approx} \lim_{h\to 0} \frac{C(q+h)-C(q)}{(q+h)-q} = \frac{dC}{dq} = M.C.(q)$$

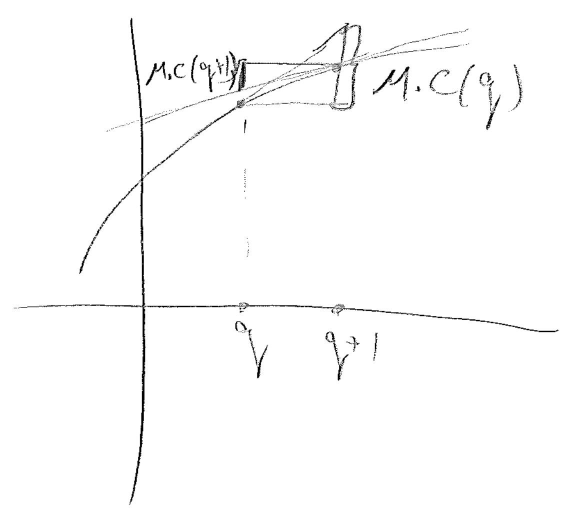
Remarks: ① Similarly we define:

$M.R.(q) = \frac{dR}{dq}$ ← Marginal Revenue $\quad R_{1\,more}(q) = R(q+1) - R(q)$

$M.P.(q) = \frac{dP}{dq}$ ← Marginal profit $\quad P_{1\,more}(q) = P(q+1) - P(q)$

② Why use $M.C.(q)$ and not $M.C.(q+1)$?

Actually, this was arbitrary and both are valid answers for "large" q it usually doesn't matter much (we are approx. any way).



If any time left:

Proof that $\frac{d \arctan(x)}{dx} = \frac{1}{1+x^2}$:

$y = \arctan(x)$

$\tan(y) = x$

$\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x) = 1$

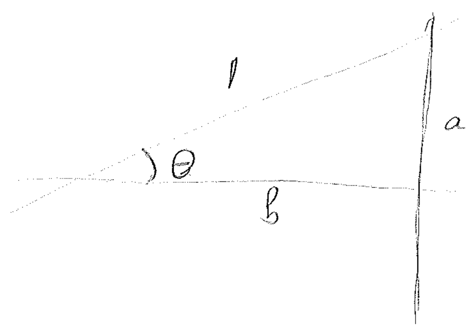
$\quad\; \parallel$

$\frac{1}{\cos^2(y)} \cdot y'$

So: $y' = \cos^2(y)$ But what is $\cos(\arctan(x))$?

assume $x > 0$ (HW: What if $x < 0$?)

$x = \frac{a}{b}$



$\tan(\theta) = \frac{a}{b} = x$

$\theta = \arctan(x)$

$\cos(\arctan(x)) = \cos(\theta) = \frac{b}{1} = b.$

Since $a^2 + b^2 = 1$ we have: $x = \frac{a}{b} = \frac{\sqrt{1-b^2}}{b}$

or, in other words: $b^2 x^2 = 1 - b^2$

$b^2(1+x^2) = 1$

$b^2 = \frac{1}{1+x^2}$

$b = \frac{1}{\sqrt{1+x^2}}$ so $\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$

$y' = \frac{1}{\ldots}$

Goals: - Average Marginal Cost
- Rates of change
- Midterm Solutions

Exercise:

It costs a small firm $C(q)$ dollars to produce q tables

where $C(q) = 100 + 50q - \frac{1}{50}q^2$

① If 500 tables are manufactured, then the fixed costs are?

\$100

② The average cost is? $\frac{C(500)}{500} = \frac{100 + 50\cdot 500 - \frac{1}{50}\cdot 500^2}{500} = \$40.1$

③ The Marginal Unit Cost (MUC), i.e $C_{1\,more}(500)$, is?

$MUC(500) = C(501) - C(500) = \left[100 + 50\cdot 501 - \frac{1}{50}(501)^2\right] - \left[100 + 50\cdot 500 - \frac{1}{50}\cdot(500)^2\right] =$

$= 50(501 - 500) - \frac{1}{50}(501-500)(501+500) =$

$= 50 - \frac{1}{50}\cdot 1001 = \$29.98$

④ The marginal cost is? $MC(q) = C'(q) = 50 - \frac{1}{25}q$

$C'(500) = 50 - \frac{500}{25} = 50 - 20 = \$30$

⑤ $MC(800) = ?$ $MC(800) = 50 - \frac{800}{25} = \$18$

Suppose the total revenue function for a commodity is

$$R(q) = 30q - 0.07q^2$$

(7) What is the unit selling price when $q = 50$?

$$\text{price}(50) = \frac{R(50)}{50} = \frac{30 \cdot 50 - 0.07 \cdot 50^2}{50} = 30 - 3.5 = \$26.5$$

$$\boxed{\text{Price} = \frac{R(q)}{q}}$$

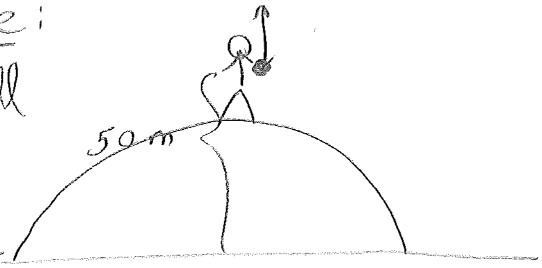
(8) The marginal revenue of production at 50 units is?

$$MR(q) = R'(q) = 30 - 0.14q$$

$$MR(50) = 30 - 0.14 \cdot 50 = 30 - 7 = \$23$$

Rates of Change:

You throw a ball in the air from a hill-top



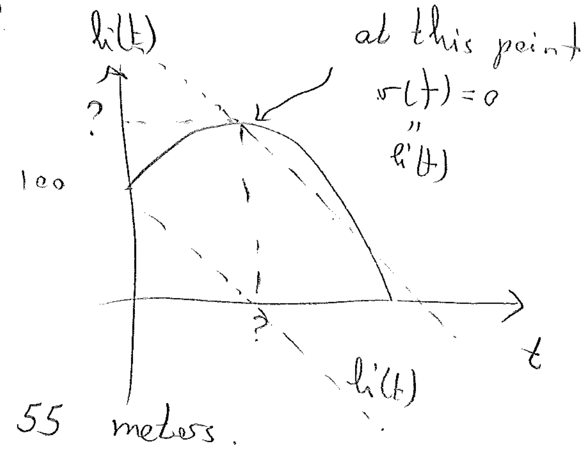
$$h(t) = -5t^2 + 10t + 50.$$

① What is the initial velocity and acceleration?

$$v = \frac{dh}{dt} = -10t + 10. \qquad v(0) = 10 \ m/sec \quad (+, \text{ i.e. up})$$

$$a = \frac{dv}{dt} = \frac{d^2h}{dt^2} = -10 \ \frac{m}{sec^2} \quad \text{always!}$$

② How high the stone go?



$$v(t) = -10t + 10 \overset{!}{=} 0$$

$$\boxed{t = 1} \qquad h(1) = -5 \cdot 1^2 + 10 \cdot 1 + 50 = 55 \text{ meters.}$$

③ How fast is the stone when it hits the ground?
Assume ground level is 49 meters.

$$h(t) = 49 \qquad -5t^2 + 10t + 1 = 0 \qquad t_0 = \frac{-10 - \sqrt{100 + 20}}{-10} = 1 + \sqrt{1.2}$$

$$v(t_0) = -10(1 + \sqrt{1.2}) + 10 = -10 \cdot \sqrt{1.2} \quad (\text{i.e. down})$$

A few examples from the Midterm:

② (b) Use the IVT to show that $\ln(x)+3^x = 1+4x$ has a solution.

Sol: (First note that $x>0$! Cannot do $f(0)$!)

· Let $f(x) = \ln(x)+3^x-1-4x$.

Note that:

· $f(1) = 0+3-1-4 = -2<0$

· $f(2) = \underbrace{\ln(2)}_{>0} + \underbrace{3^2-1-4\cdot 2}_{=0} > 0$

· $f(x)$ is cont. on $[1,2]$

By IVT there exists $1\le x\le 2$ s.t. $f(x)=0$,

i.e. $\ln(x)+3^x = 1+4x$.

④ (b) Find the value of the parameters A and B that make the following function diff. at $x=1$

$$f(x)=\begin{cases} Ax+B, & x<1 \\ \sqrt{x}, & x\ge 1\end{cases}$$

Sol: First we need to check cont.

$$\lim_{x\to 1^-} f(x) = A+B, \quad \lim_{x\to 1^+} f(x) = \sqrt{1} = 1$$

So $A+B=1$ (I)

on the other hand

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \left(\frac{d}{dx}\sqrt{x}\right)\Big|_{x=1} = \left(\frac{1}{2\sqrt{x}}\right)\Big|_{x=1} = \frac{1}{2}$$

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \frac{d}{dx}(Ax+B)\Big|_{x=1} = A$$

So $A = \frac{1}{2}$

and $B = 1-\frac{1}{2} = \frac{1}{2}$

① ⓒ How much money is required to generate $8,000 in interest over one year if the interest rate is 6% compounded continuously?

Sol: $\begin{cases} FV = PV + 8{,}000 \\ FV = PV \cdot e^{rt} = PVe^{0.06} \end{cases}$ $r = 0.06, t = 1$

So: $PV + 8000 = PV \cdot e^{0.06}$ hence $\boxed{PV = \frac{8000}{e^{0.06} - 1}}$

③ ⓑ Given that $f(1) = 2$, $f'(1) = -1$ and $p(x) = \frac{x^2+1}{f(x)}$

What is the tangent line to the curve $y = p(x)$ at $x = 1$?

Sol: • $p(1) = \frac{1^2+1}{f(1)} = \frac{1+1}{2} = 1$

• $p'(x) = \frac{2x \cdot f(x) - (x^2+1) f'(x)}{f(x)^2}$ So $p'(1) = \frac{2 \cdot 2 - 2 \cdot (-1)}{2^2} = \frac{6}{4} = \frac{3}{2}$

$l(x) = p'(1) \cdot (x-1) + 1 = \frac{3}{2}(x-1) + 1 = \frac{3}{2}x - \frac{1}{2}$.

Relative Rates of Change:

Q1: Suppose that 10% increase in price would create a 10% decrease in ~~revenue~~ quantity demand. What would be the effect on the revenue?

Increase, decrease, stay the same, can't say.

$R_{new} = (1.01p)\cdot(0.99) = (1.01\cdot 0.9)(p\cdot q) = 0.909\ R_{old} < R_{old}$

Q2: Your CFO (Chief Financial Officer) reports that profits increased by \$3,000 last year. Is that good? Yes/No/Maybe/other?

RBC Profits at \$9.27 B

BES Build/Design at \$15.7 M

So it depends.

Q3: Last year a pair of socks increased from \$8 to \$10 and a truck increased from \$20,000 to \$21,000. Did the price of the truck increase 500 times that of a pair of socks?

Def: If $f(t)$ is the price of an item at time t the the $\underbrace{\text{rate}}_{\text{(absolute)}}$ of change is $f'(t)$ and the relative rate of change is $\dfrac{f'(t)}{f(t)}$

In the example:

$$\frac{21{,}000 - 20{,}000}{20{,}000} = 0.05$$

$$\frac{\$10 - \$8}{8} = 0.25.$$