Extreme values:

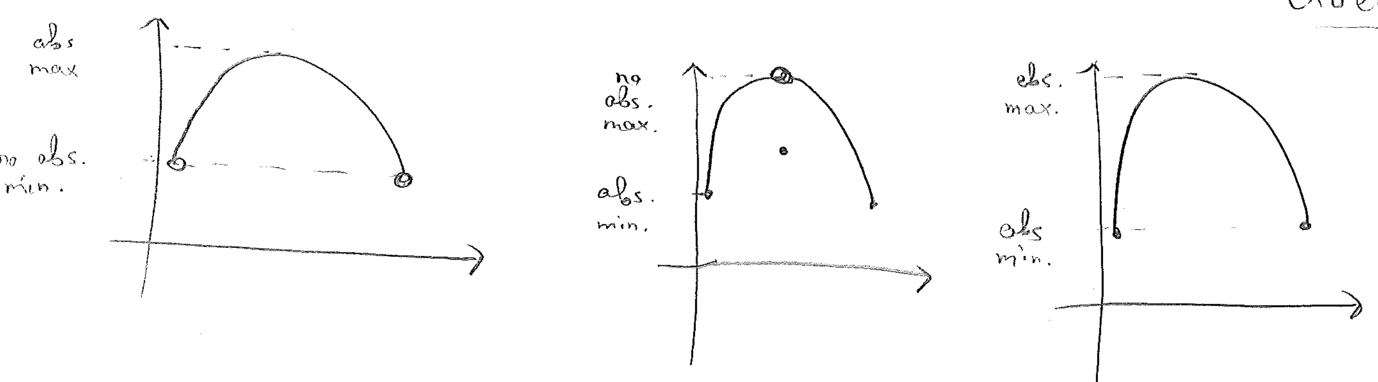
Let f be defined on an interval I containing c.

① $f(c)$ is the <u>absolute minimum</u> of f on I
if $f(c) \le f(x)$ for all x in I.

② $f(c)$ is the <u>absolute maximum</u> of f on I
if $f(c) \ge f(x)$ for all x in I.

$f(c)$ is called <u>extreme value</u> and c an <u>extreme point</u>

<u>Examples</u>:



<u>Extreme Value Theorem (EVT)</u>:

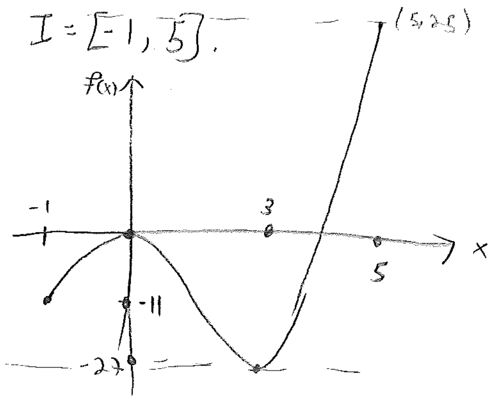
Let f be a continuous function defined on a closed interval I. Then f has both an absolute maximum and minimum value on I.

<u>Q</u>: How can we find them? (Say, we want to maximize profit)

<u>Example</u>: Consider $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$.

<u>Idea of Solution</u>:

Sketch of $f(x)$:



Seems like the absolute extremums are $(3,-27)$, $(5,25)$. But how can we be sure? (Do we <u>know</u> how to sketch?)

Answer: Next time.

Local / Relative Minimum / Maximum:

Let f be defined on an interval I containing c.

① If there is an open interval J containing c (and contained in I) such that $f(c) \le f(x)$ for any x in J. then we call $f(c)$ a <u>local minimum</u>.

② If there is an open interval J containing c (and contained in I) such that $f(c) \ge f(x)$ for any x in J then we call $f(c)$ a <u>local maximum</u>.

$f(c)$ is called a <u>local extreme value</u>
c is called a <u>local extreme point</u> / local extrema

<u>In the example</u>: $(0,0)$ - local maximum
$(3,-27)$ - local minimum

$(-1,-11)$ and $(5,25)$ are <u>not</u> local extreme points.
Because no open intervals around $-1, 5$ in $[-1,5]$

<u>Q</u>: Can we find these?
We already know that $f'(c) > 0 \rightsquigarrow$ "f increasing at c"
$f'(c) < 0 \rightsquigarrow$ "f decreasing at c"

<u>Fermat's Theorem</u>: Assume $f(x)$ is differentiable at c and $(c, f(c))$ is a local extrema of f then $f'(c) = 0$.

<u>Def</u>: Let f be defined near c. We say that c is a <u>critical point</u> of f ($f(c)$ a critical number) if $f'(c) = 0$ or $f'(c)$ is not defined.

Examples:

① $f(x) = 2x^3 - 9x^2$

$f'(x) = 6x^2 - 18x$

$f'(c) = 0 \qquad 6c^2 - 18c = 0 \qquad c^2 - 3c = 0 \qquad c(c-3) = 0$

Critical points? $\qquad c = 0 \qquad c = 3$

How can we tell if this is a local extrema? and which?

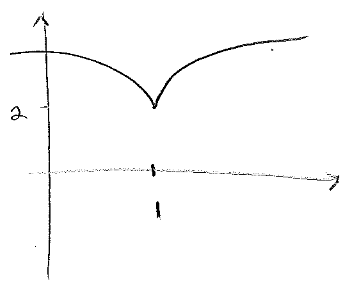
$c = -1$ and $c = 5$ are not!

② $f(x) = (x-1)^{2/3} + 2$ $\quad |x-1|^{2/3}$

$f'(x) = \frac{2}{3}(x-1)^{-1/3}$

$f'(c) = 0$? Doesn't happen.

However $f'(1)$ doesn't exist. So $c = 1$ is a critical point.

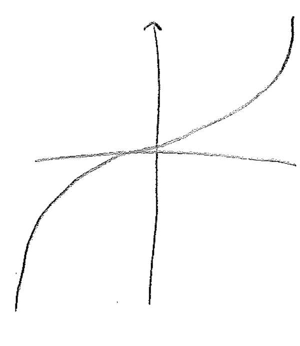


(actually, global minima)
(and local)

③ Non-example: $f(x) = x^3$

$f'(x) = 3x^2$

$f'(c) = 0 \longrightarrow c = 0$



$c = 0$ is a critical point

but it is not a local extrema.

Let's go back to $f(x) = 2x^3 - 9x^2$. on an interval. $[-1, 5]$.

We have critical points $x = 0, 3$

Any absolute maxima or minima of f in $[-1, 5]$ is either a local extrema on an edge.

So there are only 4 candidates $-1, 0, 3, 5$

$f(-1) = -11$

$f(0) = 0$

$f(3) = -27 \leftarrow$ abs. ~~max~~. min.

$f(5) = 25 \longleftarrow$ abs max.

Closed Interval Method

Let: f be a continuous function on $[a,b]$.

EVT: f has an absolute $\left(\text{global}^{\text{also called}}\right)$ maxima and minima in $[a,b]$
(Weierstrass)

The Idea: ① If c is a global extremum of f in $[a,b]$ then either c is a local extremum of f or c is an edge ($c=a$ or b)

② If c is a local extremum point then c is a critical point of f. namely $f'(c)=0$ or $f'(c)$ doesn't exist.

Simple Algorithem: ① Find critical points $\left(\begin{array}{l}\text{solve } f'(c)=0 \text{ and find points} \\ \text{where } f'(c) \text{ is not defined}\end{array}\right)$

For most examples you'll encounter this is a finite list $c_1, \dots, c_n$

② Compute values at critical points and edges.

③ Compare.

Examples:

① Find extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0,3]$

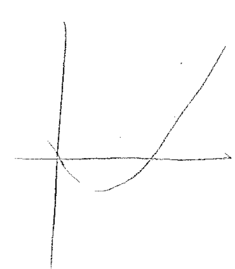
Sol: $f'(x) = 6x^2 + 6x - 12$

$f'(c) = 0 \qquad \cancel{6}c^2 + \cancel{6}c - \cancel{12}^{2} = 0 \qquad c^2 + c - 2 = 0 \qquad c = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$

So $c=1$ is the only critical point in $[0,3]$

$$\begin{cases} f(0) = 0 \\ f(1) = -7 \\ f(3) = 45 \end{cases}$$

So: $\boxed{\begin{array}{lll} \text{Abs. Max:} & 45 \text{ at} & x=3 \\ \text{Abs. Min.} & -7 \text{ at} & x=1 \end{array}}$



Example: (2) $f(x) = \begin{cases} (x-1)^2, & x \le 0 \\ x+1, & x > 0 \end{cases}$ find extreme values on $[-4,2]$

Sol: $\lim_{x \to 0^+} f(x) = 1 = \lim_{x \to 0^-} f(x)$ so $f(x)$ is cont. at 0 (and actually at $[-4,2]$ so EVT applies)

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\overset{x^2-2x+1}{(x-1)^2 - 1}}{x - 0} = \lim_{x \to 0} \frac{\overset{x-2}{\cancel{x^2 - 2x}}}{\cancel{x}} = -2$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(x+1) - 1}{x - 0} = 1$$ So f is not diff. at $x=0$

$c=0$ is a critical point.

For $x<0$: $f'(x) = 2(x-1)$; $f'(c)=0$ implies $c=1 \not< 0$

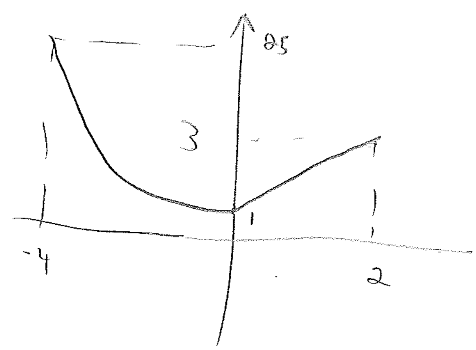
so no critical point $c<0$.

For $x>0$: $f'(x) = 1 \neq 0$. Critical points: $c=0$

$f(-4) = 25$ ⟵ Abs Max.

$f(0) = 1$ ⟵ Abs. Min

$f(2) = 3$

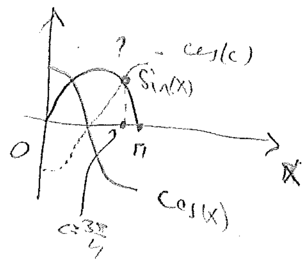


③ $f(x) = e^x \overset{>0}{\sin x}$ on $[0,\pi]$

Sol: $f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$.

$f'(c) = 0$ implies $\sin(c) + \cos(c) = 0$

$\longrightarrow \boxed{c = \frac{3\pi}{4}}$ (only solution in $[0,\pi]$)



$f(0) = 0$
$f(\pi) = 0$
$f(c) = e^{\frac{3\pi}{4}} \overbrace{\sin\left(\frac{3\pi}{4}\right)}^{\frac{\sqrt{2}}{2}} > 0$ So:

Abs min: 0 at $x=0$ or $x=\pi$
Abs max: $\frac{\sqrt{2}}{2} e^{\frac{3\pi}{4}}$ at $x = \frac{3\pi}{4}$

④ $f(x) = \frac{\overset{>0}{\ln(x)}}{x}$ on $[1,4]$

Sol: $f'(x) = \frac{\frac{1}{x} \cdot x - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$

$f'(c) = 0$ implies $\ln(c) = 1$, i.e $1 < c = e < 4$

$f(1) = 0$ ⟵ Abs min
$f(e) = 1/e \approx 0.36..$ ⟵ Abs max.
$f(4) = \frac{\ln(4)}{4} \approx 0.34..$