Local Extrema points:

We want to be able to find local min./max. of functions.

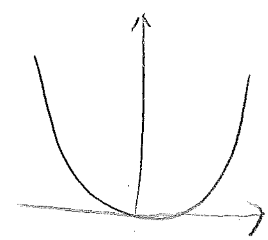
Def: Let f be a function defined on an interval I.

(f is called monotonous)

① f is increasing on I if for every $a<b$ in I, $f(a) \le f(b)$
(strictly increasing if $a<b$ implies $f(a) < f(b)$)

② f is decreasing on I if for every $a<b$ in I, $f(a) \ge f(b)$
(strictly decreasing if $a<b$ implies $f(a) > f(b)$)

Examples: ① $f(x) = x^2$ is strictly increasing on $[0,\infty)$
and strictly decreasing on $(-\infty, 0]$ and
not increasing nor decreasing on $[-1,1]$



② Food for thought: If f is both increasing and decreasing then f is constant.

Before we go into more complicated examples, how can we tell when a function is increasing/decreasing?

Thm: Let f be a continuous function on $[a,b]$ and diff. on (a,b)

① If $f'(c) > 0$ for all c in (a,b) then f is increasing on $[a,b]$

② If $f'(c) < 0$ ———— // ———— decreasing on $[a,b]$

③ If $f'(c) = 0$ ———— // ———— constant on $[a,b]$

Examples ①

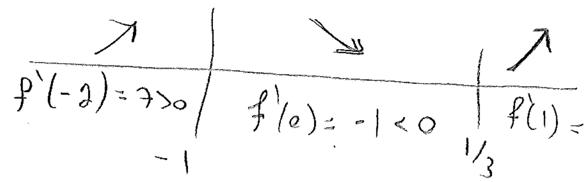
1. $f(x) = x^3 + x^2 - x + 1$, Find the biggest intervals on which f is increasing or decreasing.

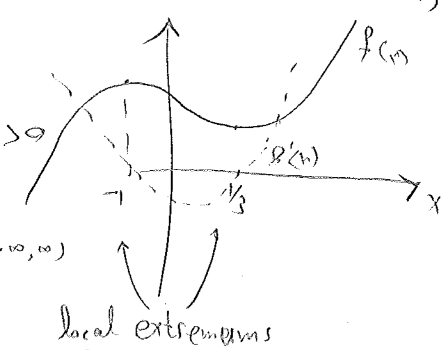
Sol: $f'(x) = 3x^2 + 2x - 1$.

$f'(x)$ is cont. and so, in order to change sign it must go through 0.

$f'(c) = 0 \longrightarrow 3c^2 + 2c - 1 = 0 \longrightarrow c = \frac{-2 \pm \sqrt{4+12}}{6} = \frac{-2 \pm 4}{6} = -1, \frac{1}{3}$

enough to check on value in each interval $(-\infty,-1]$, $[1, \frac{1}{3}]$, $[\frac{1}{3}, \infty)$





② $f(x) = \sqrt[3]{x}$, $f'(x) = \frac{1}{3x^{2/3}} > 0$ for all $x \neq 0$, yet $f(x)$ is increasing on $(-\infty, \infty)$

Theorem (First derivative test)

Let f be a diff. function on I and let c be a critical point in I

① If the sign of f' switch from positive to negative at c, then $f(c)$ is a local maximum. ~~point~~

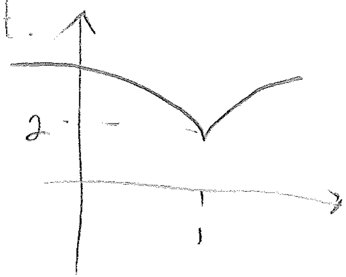
② If the sign of f' switch from negative to positive at c then $f(c)$ is a local minimum. ~~point~~

③ If the sign of f' does not change at c then $f(c)$ is not a local extrem

Examples: ① Find local extrema of $f(x) = (x-1)^{2/3} + 2$.

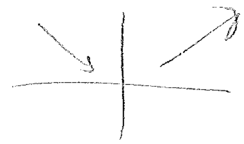
Sol: $f'(x) = \dfrac{2}{3\sqrt[3]{x-1}}$ $x \neq 1$, no solution for $f'(c) = 0$

So $c = 1$ is the only critical point.



$f'(9) = \dfrac{2}{3\sqrt[3]{8}} = \dfrac{1}{3} > 0$

$f'(-7) = \dfrac{2}{3\sqrt[3]{-8}} = -\dfrac{1}{3} < 0$



Local minima.

② Find local extrema of $f(x) = \dfrac{x^2+3}{x-1}$

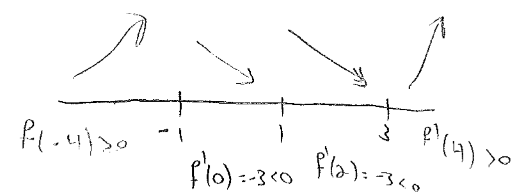
Sol: · Critical points: $f'(x) = \dfrac{2x(x-1) - (x^2+3)}{(x-1)^2}$ $x \neq 1$

($2x(x-1) = 2x^2 - 2x$, $-(x^2+3) = -x^2 - 3$)

$= \dfrac{x^2 - 2x - 3}{(x-1)^2}$

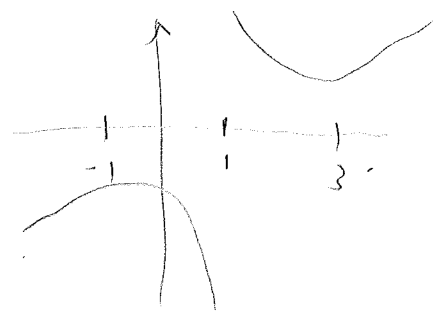
$f'(c) = 0 \longrightarrow c^2 - 2c - 3 = 0 \longrightarrow c = \dfrac{2 \pm \sqrt{4+12}}{2} = \dfrac{2 \pm 4}{2}$ → 3, −1

critical points: $c = -1, 1, 3$.



−1 is a local max.

3 is a local min.



$x^2 + 3 > 0$ for any x

$\begin{cases} x - 1 > 0 & \text{for } x > 1 \\ x - 1 < 0 & \text{for } x < 1 \end{cases}$

Information In the Second Derivative

Concavity:

Let f be diff. on an interval I. The graph of f is <u>concave up</u> on I if f' is increasing on I. The graph of f is <u>concave down</u> on I if f' is decreasing on I.

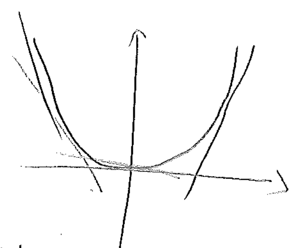
<u>Remark</u>: f is concave up & down if and only if it is linear / f' cons

Sometimes called no-concavity.

What does it mean?

<u>Examples</u>: ① $f(x) = x^2$

$f'(x) = 2x$ increasing



concave up.

tangents are under <u>the graph</u>

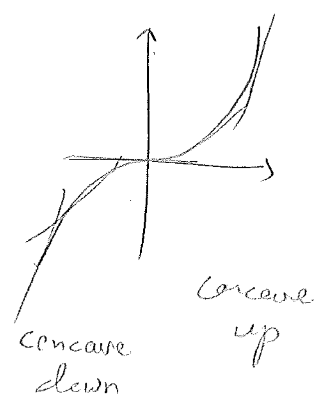
② $f(x) = x^3$

$f'(x) = 3x^2$ increasing for $x \geq 0$

decreasing for $x \leq 0$

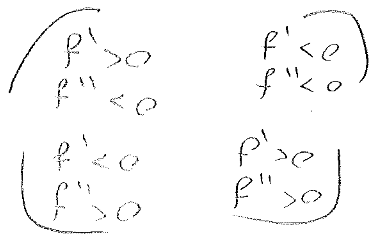
Concave up = tangents under graph

Concave down = tangents ~~over~~ above graph



Theorem (Test for concavity):

Let f be twice diff. (f' is also 'diff') on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.



Def: A point of inflection of f is a point at which the concavity of f changes.

Thm: If $(c, f(c))$ is a point of inflection for f then either $f''=0$ or f'' is not defined at c.

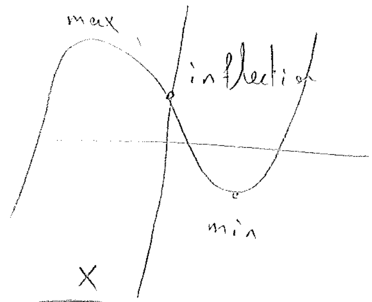
Examples:

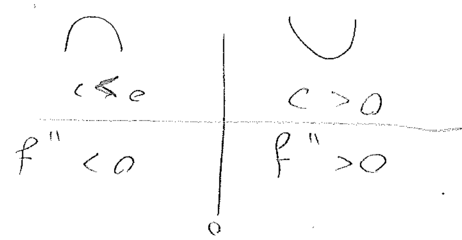
① Find points of inflection and intervals of concavity for $f(x) = x^3 - 3x + 1$

Sol: $f'(x) = 3x^2 - 3$
$f''(x) = 6x$

$f''(x)$ defined for all x

$f''(c) = 0 \longrightarrow c = 0$



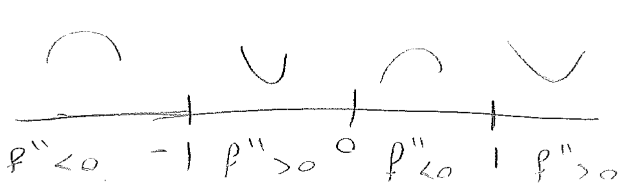


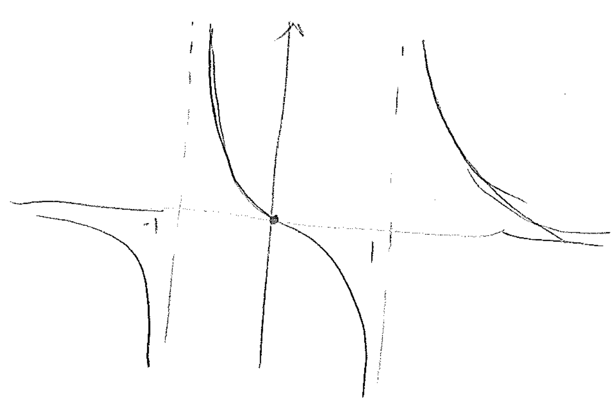
② $f(x) = \dfrac{x}{x^2-1}$

$$f'(x) = \frac{1\cdot(x^2-1) - 2x\cdot x}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

$$f''(x) = -\frac{2x(x^2-1)^2 - 2(x^2-1)\cdot 2x(x^2+1)}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3} \quad \text{for all } x \neq \pm 1$$

$f''(c) = 0 \longrightarrow 2c\cdot\underbrace{(c^2+3)}_{>0} = 0 \longrightarrow c = 0$





all three are pts of inflection.

Remark: The point of inflection is a critical point for f'.

Price elasticity of demand - Review

$R(p) = p \cdot q(p)$

$$\frac{dR}{dp} = q + p \cdot \frac{dq}{dp} = q\Big[1 + \underbrace{\frac{p}{q}\frac{dq}{dp}}_{\varepsilon(p)}\Big]$$

$\varepsilon(p)$ called demand elasticity

The law of demand implies that $\frac{dp}{dq} < 0$.

We care about the sign of $1+\varepsilon(p)$.

① $|\varepsilon| > 1$ ($\varepsilon < -1$, $1+\varepsilon < 0$) we say the good is price elastic and $\frac{dR}{dp} < 0$: decrease price for higher revenue.

② $|\varepsilon| < 1$ ($-1 < \varepsilon < 0$, $0 < 1+\varepsilon$); we say the good is price in-elastic

$\frac{dR}{dp} > 0$: increase price for higher revenue.

③ $|\varepsilon| = 1$ ($\varepsilon = -1$, $1+\varepsilon = 0$); we say the good is price unit elasti[c]

$\frac{dR}{dp} = 0$: optimal price, revenue is (local) maximal.

Examples:

① The price p (in \$) and the demand q for a product are related by $p^2 + 2q^2 = 1{,}100$. If the current price per unit is \$30, will revenue increase or decrease if the price is slightly raised?

$30^2 + 2q^2 = 1100 \to q = 10$

Sol: $p^2 + 2q^2 = 1{,}100 \to 2p + 4q\frac{dq}{dp} = 0 \longrightarrow \frac{dq}{dp} = -\frac{2p}{4q} = -\frac{p}{2q}$

$p = 30,\ q = 10$

$\varepsilon = \frac{p}{q}\frac{dq}{dp} = -\frac{p^2}{2q^2} = -\frac{9}{2} < -1 \leftarrow$ price elastic $\longrightarrow$ revenue decrease

Elasticity Review:

② A cell phone supplier has determined that demand for its newest cell phone model is given by

$$qp + 30p + 50q = 8{,}500$$

q - quantity of units sold
p - price in \$.

If the current price is \$150, will revenue increase or decrease if the price is slightly lowered? What is the optimal price?

Sol: * $\frac{dR}{dp} = q + p\frac{dq}{dp} + 30 + 50\frac{dq}{dp} = 0$

$$\frac{dq}{dp} = -\frac{30+q}{50+p} \qquad \varepsilon = \frac{p}{q}\frac{dq}{dp} = -\frac{p(30+q)}{q(50+p)}$$

$\frac{150 \cdot 50}{20 \cdot 50}$

$p = 150 \rightsquigarrow 150q + 30 \cdot 150 + 50q = 8{,}500 \longrightarrow \boxed{q = 20}$

$\varepsilon(150) = -\frac{15}{8} < -1$ ← elastic. revenue increased if price decreased

$-\frac{150 \cdot 50}{20 \cdot 200}$

* We try to solve $\varepsilon = -1$ to find the optimal price.

$\cancel{pq} + 50q = \cancel{pq} + 30p \Longrightarrow 3p = 5q \quad q = \frac{3}{5}p, \; p = \frac{5}{3}q$

$qp + 60p = 8{,}500 \longrightarrow \frac{5}{3}q \cdot p + 100p = \frac{5}{3} \cdot 8{,}500$

$p^2 + 100p - \frac{5}{3}8{,}500 = 0$

$p = \frac{-100 \pm \sqrt{\ldots}}{2} \approx \$79$

The Second Derivative Test:

Thm: Let c be a critical point of f where $f''(c)$ is defined.

① If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.

② If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

Rem: If $f''(c) = 0$, the test is inconclusive

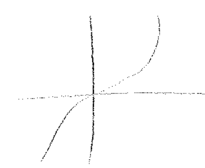
Examples:

① $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$

$f'(c) = 0 \rightarrow c = 0$. $f''(0) = 0$

But $f'(x) \geq 0$ for all x,

so not a local extremum.



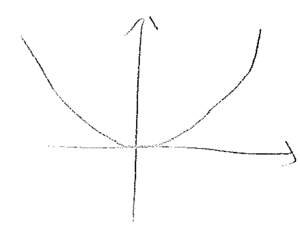
② $f(x) = x^4 = f'(x) = 4x^3$ $f''(x) = 12x^2$

$f'(c) = 0 \rightarrow c = 0$, $f''(0) = 0$

But $f'(x) > 0$ for $x > 0$

$f'(x) < 0$ for $x < 0$

So $x = 0$ is a local minimum.



min → concave up

max → concave down.

③ $f(x) = \frac{1}{y} + x$

$f'(x) = -\frac{1}{x^2} + 1$ $f'(c) = 0 \rightarrow \frac{c^2+1}{c^2} = 0$ $c = \pm 1$

$f''(x) = \frac{2}{c^3}$ $f''(\pm 1) = \pm 2$ so $+1$ local max

-1 local min