Math 104 section 108 Quiz #2 Column A   Date: ____________ Grade: ____________

First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Short answer questions — you must show your work

1. $\boxed{\text{6 marks}}$ Each part is worth 2 marks.

(a) Prove that the equation $\ln x = \frac{1}{x}$ has a solution.

Solution: We define $f(x) = \ln(x) - \frac{1}{x}$ for $x > 0$. Since

1. $f(1) = \ln 1 - 1 = 0 - 1 = -1 < 0$ and
2. $f(e) = \ln e - \frac{1}{e} = 1 - \frac{1}{e} > 0$ and
3. f is continuous on $[1, e]$

then IVT implies the existence of $1 < c < e$ such that $f(c) = 0$.

(b) Differentiate the function $y = \frac{e^{x^3} - x^2}{\ln x}$. You do not need to simplify your answer. You may use any method taught in class.

Answer:
$y' = \frac{(3x^2e^{x^3} - 2x)\ln x - \frac{e^{x^3} - x^2}{x}}{\ln^2 x}$

Solution:

$$\begin{aligned} y' &= \frac{(e^{x^3} - x^2)' \cdot (\ln x) - (e^{x^3} - x^2) \cdot (\ln x)'}{\ln^2 x} \\ &= \frac{(e^{x^3} \cdot (x^3)' - 2x) \cdot (\ln x) - (e^{x^3} - x^2) \cdot \frac{1}{x}}{\ln^2 x} \\ &= \frac{(3x^2e^{x^3} - 2x)\ln x - \frac{e^{x^3} - x^2}{x}}{\ln^2 x} \end{aligned}$$

(c) Write the line equation for the tangent of $f(x) = x^a$ at $x = 1$. Find a value a for which this tangent passes through the point $(0, 0)$.

Answer: $l(x) = a(x-1)+1$, $a = 1$

Solution: We have $f(1) = 1$. The tangent formula is $l(x) = f'(1)(x - 1) + 1$. We have $f'(x) = ax^{a-1}$ and hence $f'(1) = a$. Hence $l(x) = a(x - 1) + 1$.

For the second part we plug in $x = 0$ and $l = 0$ to get $0 = a(0 - 1) + 1$ and hence $a = 1$.

Long answer question — you must show your work

2. $\boxed{\text{4 marks}}$ Compute the derivative of $f(x) = \sqrt{x^2+5}$ at $x = 2$ and write the line equation of the tangent line. You may only use the definition of the derivative.

Answer: $f'(3) = \frac{2}{3}$, $l(x) = \frac{2}{3}(x - 2) + 3$

Solution:

$$\begin{aligned}
f(2) &= \sqrt{2^2+5} = \sqrt{9} = 3 \\
f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} \\
&= \lim_{x\to 2} \frac{\sqrt{x^2+5} - 3}{x-2} \\
&= \lim_{x\to 2} \frac{(\sqrt{x^2+5} - 3)(\sqrt{x^2+5} + 3)}{(x-2)(\sqrt{x^2+5} + 3)} \\
&= \lim_{x\to 2} \frac{(x^2+5) - 3^2}{(x-2)(\sqrt{x^2+5} + 3)} \\
&= \lim_{x\to 2} \frac{x^2 - 4}{(x-2)(\sqrt{x^2+5} + 3)} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+5} + 3)} \\
&= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+5} + 3} = \frac{2+2}{\sqrt{2^2+5} + 3} = \frac{4}{6} = \frac{2}{3} \\
l(x) &= \frac{2}{3}(x-2) + 3
\end{aligned}$$