Math 104 section 108 Quiz #2 Column B   Date: Grade:

First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

Short answer questions — you must show your work

1. 6 marks Each part is worth 2 marks.

   (a) Prove that the equation $e^x = \frac{1}{x}$ has a solution.

   Solution: We define $f(x) = e^x - \frac{1}{x}$ for $x > 0$. Since

   1. $f(1/3) = e^{1/3} - \frac{1}{1/3} = \sqrt[3]{e} - 3 < 0$ and
   2. $f(1) = e^1 - \frac{1}{1} = e - 1 > 0$ and
   3. f is continuous on $[1/3, 1]$

   then IVT implies the existence of $1/3 < c < 1$ such that $f(c) = 0$.

   (b) Differentiate the function $y = \frac{e^{x^4} - x^3}{\ln x}$. You do not need to simplify your answer. You may use any method taught in class.

   Answer: $y' = \frac{(4x^3e^{x^4} - 3x^2)\ln x - (e^{x^4} - x^3)\frac{1}{x}}{\ln^2 x}$

   Solution:

   $$\begin{aligned} y' &= \frac{(e^{x^4} - x^3)' \ln x - e^{x^4} - x^3 (\ln x)'}{\ln^2 x} \\ &= \frac{(4x^3 e^{x^4} - 3x^2)\ln x - (e^{x^4} - x^3)\frac{1}{x}}{\ln^2 x} \end{aligned}$$

   (c) Write the line equation for the tangent of $f(x) = ax^2 - ax$ at $x = 1$. Find a value a for which this tangent passes through the point $(0, 0)$.

   Answer: $l(x) = a(x-1)+1$, $a = 1$

   Solution: We have $f(1) = 0$. The tangent formula is $l(x) = f'(1)(x-1)$. We have $f'(x) = 2ax - a = a$ and hence $f'(1) = a$. Hence $l(x) = a \ln a(x-1) + 1$.
   For the second part we plug in $x = 0$ and $l = 0$ to get $0 = a(0-1)$ and hence $a = 0$.

Long answer question — you must show your work

2. 4 marks Compute the derivative of $f(x) = \sqrt{x^2 + 7}$ at $x = 3$ and write the line equation of the tangent line. You may only use the definition of the derivative.

Answer: $f'(3) = \frac{3}{4}$, $l(x) = \frac{3}{4}(x - 3) + 4$

Solution:

$$\begin{aligned}
f(3) &= \sqrt{3^2+7} = \sqrt{16} = 4 \\
f'(3) &= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \\
&= \lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3} \\
&= \lim_{x\to 3} \frac{(\sqrt{x^2+7}-4)(\sqrt{x^2+7}+4)}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{(x^2+7)-4^2}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{x^2-9}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{x+3}{\sqrt{x^2+7}+4} = \frac{3+3}{\sqrt{3^2+7}+4} = \frac{6}{8} = \frac{3}{4} \\
l(x) &= \frac{3}{4}(x-3)+4
\end{aligned}$$