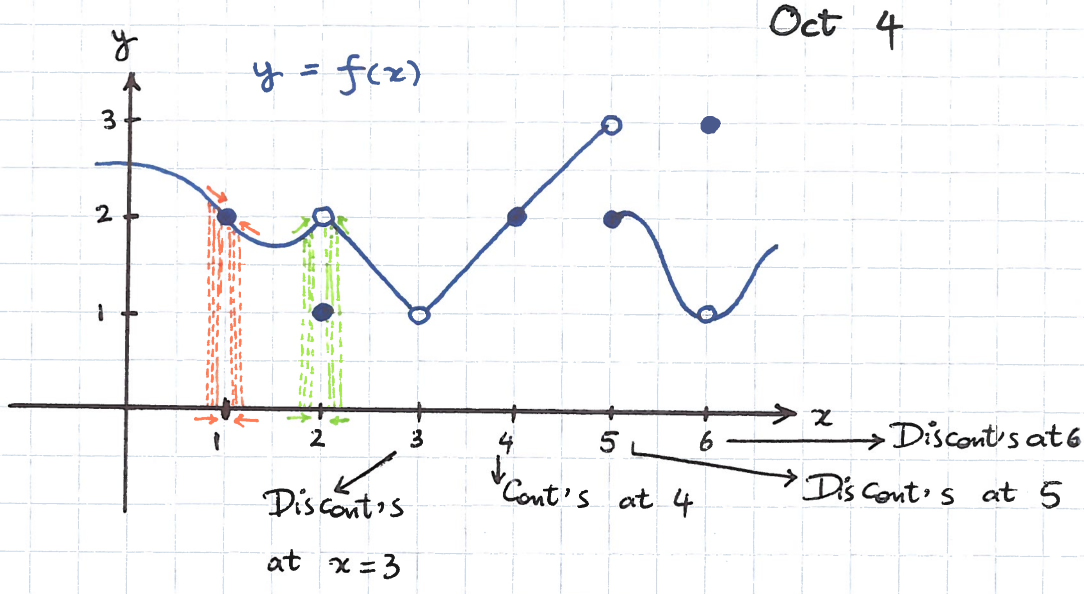
"Continuous Functions"

Lecture 8
Oct 4



We have already seen that it is possible to have a function $f(x)$ such that

$$\lim_{x \to a} f(x) = L \quad \text{BUT} \quad f(a) \neq L .$$

For example, in the graph above for

$x = 2$: $\underbrace{\lim_{x \to 2} f(x) = 2 \quad \text{BUT} \quad f(2) = 1}_{\neq}$

There are also cases that the value of the limit and the function are equal. From the graph above, for

$x = 1$: $\underbrace{\lim_{x \to 1} f(x) = 2 \quad \text{and} \quad f(1) = 2}_{=}$

Focusing on this property and the relation between limit and its value, we have the following definition:

Definition. Continuous Function at $x = a$

A Function $f(x)$ is said to be continuous at $x = a$ if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Hidden Facts

(I) limit of $f(x)$ must exist.

fails if

→ right lim ≠ left lim

Or

→ lim is NOT finite $= \pm\infty$

(II) $f(x)$ must be defined at a ("a" is in the domain of f)

fails if

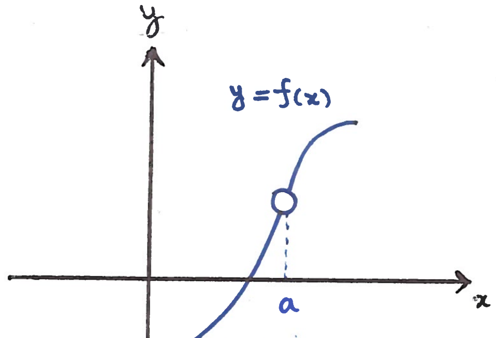
→ "a" is NOT in domain

(III) two sides are equal.

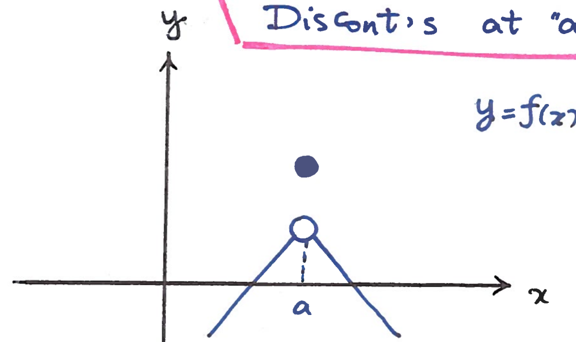
If each of the three facts (I), (II) or (III) fails, the function is discontinuous at $x = a$.

Graphical interpretation : As we move along the graph, it stays connected. You do NOT lift your pen off the paper.

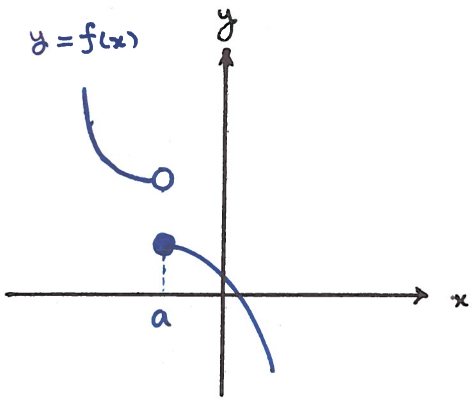
$\boxed{\text{JUMP or HOLE at } x=a \text{ Discont's at "a".}}$



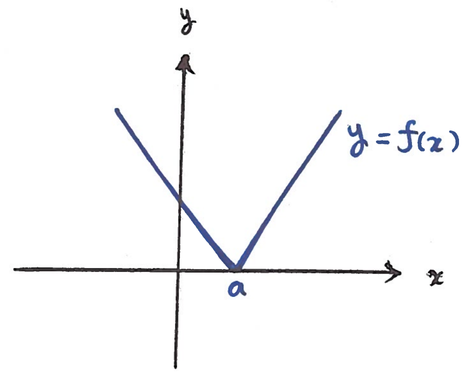
Discont's at "a" because f(a) is NOT defined.



Discont's at "a" because $\lim_{x \to a} f(x) \neq f(a)$



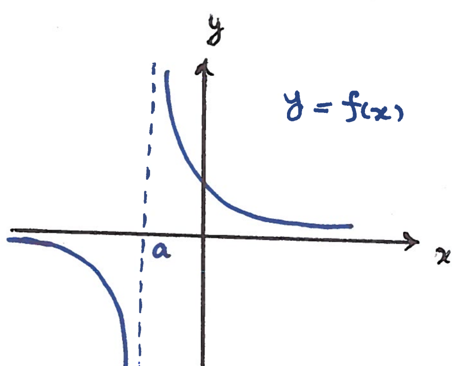
Discont's at "a" because left lim ≠ right lim so limit DNE.



Cont's at $x = a$

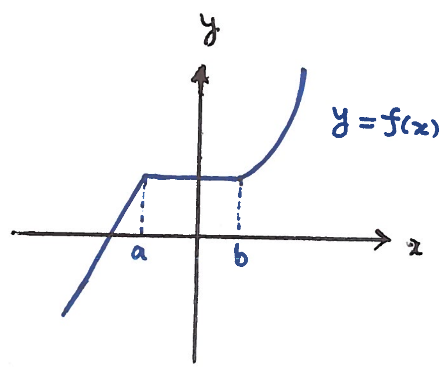
NO hole, NO Jump,

⊛ Corners are OK!



Discont's at "a" because the lim is NOT finite.

$\lim_{x \to a^-} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = +\infty$



Cont's at "a", even if it's piecewise.

NO hole, NO Jump.

Also Cont's at "b".

Check the Continuity algebraically (equation of the function)

$$\lim_{x \to a} f(x) = f(a) \quad (*)$$

Example 1. Is $f(x) = 3x^2 - 2x$ Cont's at $x = -2$?

check $(*)$:

$$\lim_{x \to -2} 3x^2 - 2x = 3(\cancel{-2})^{2} - 2\cdot(-2) = 12 + 4 = 16$$

First step of lim = substitution

$$f(-2) \overset{\text{sub}}{=} 3(-2)^2 - 2\cdot(-2) = 16$$

$\Longrightarrow$ f is cont's at $x = -2$

Example 2. Is $g(t) = t + |t - 3|$ Cont's at

(a) $t = 1$?

(b) $t = 3$?

(a) $\lim_{t \to 1} t + |t-3| \overset{\text{sub}}{=} 1 + |\cancel{1-3}| = 1 + 2 = 3$ $|-2| = 2$

Also $g(1) = 3$

$\Longrightarrow$ g cont's at $t = 1$

(b) $\lim_{t \to 3} t + |t-3| = 3 + |\cancel{3-3}| = 3$ $|0| = 0$

Also $g(3) = 3$

$\Longrightarrow$ g cont's at $t = 3$

Question. Are there any points such that the functions above are dis cont's at ?

$\rightsquigarrow$ These are "nice" functions, for which the limit always exists and substitution always works $\rightsquigarrow$ NO Points of Dis continuity.

Domain of Continuity : The points at which the function is continuous .

Functions that are Cont's Everywhere

Domain of Continuity $= \mathbb{R} = (-\infty, +\infty)$

- Linear functions : $y = mx + b$

lines

- Quadratics : $y = ax^2 + bx + c$

parabolas

- Cubics : $y = x^3$ (or any other cubic.)

$\Rightarrow$ POLYNOMIALS are Cont's everywhere

Dom of Cont'y :

$(-\infty, +\infty)$

- Cubic Root : $y = \sqrt[3]{x}$ (all x's are acceptable.)

or any "odd" root : $\sqrt[5]{x}$, $\sqrt[7]{x}$, ...

- Family of Absolute-Value

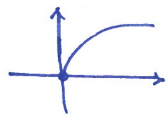
Such as : $y = |x|$ $\quad$ $y = -3 - |x+1| + |x|$

$y = |x-2| + 1$ $\quad$ $\vdots$

ignore this

Functions that are NOT Cont's everywhere.

- Root function : $y=\sqrt{x}$



$\boxed{\text{Dom of cont, } y=[0,+\infty)}$

or any other "even" root : $\sqrt[4]{x}, \sqrt[6]{x}, \ldots$

- Rational functions :

$\boxed{\text{Dom of Cont, } y = \text{Everywhere except where Denominator} = 0}$

Ex 3 . Find the domain of continuity of

(a) $f(x) = \dfrac{x}{x^2+5x+6}$

Exclude where $x^2+5x+6=0$

$\Rightarrow (x+3)(x+2)=0 \Rightarrow x=-3$ or $x=-2$ Exclude!

$\Rightarrow$ Dom of cont, y: $(-\infty,-3)\cup(-3,-2)\cup(-2,+\infty)$

(b) $h(x) = \dfrac{x+2}{x^2-4}$

$x^2-4=0 \Rightarrow (x-2)(x+2)=0 \Rightarrow x=-2$ or $x=2$ Exclude

$\Rightarrow$ Dom of Cont, y $:(-\infty,-2)\cup(-2,2)\cup(2,+\infty)$

Remark. Note that in example 3b:

$\dfrac{x+2}{(x-2)(x+2)}$, we can cancel $x+2$ only if we assume $x \neq -2$ so that to avoid $0/0$.

This implies that we still need to exclude $x=-2$.

One of the common cases of discontinuity is for piece-wise functions. For these functions, we usually have different equations to the right / left of a border point. These points are where we may have non-equal left & right limit and thus discontinuity of the function.

Ex 4. Determine the dom of continuity of

$$f(x) = \begin{cases} x^2 + 2x & x \leq -2 \rightarrow x=-2 \\ x^3 - 6x & x > -2 \end{cases}$$

$x=-2$ is the "border point", check if $\lim\limits_{x \to -2} f(x) = f(-2)$.

- $f(-2) \overset{1^{st}\text{ piece}}{=} (-2)^2 + 2\cdot(-2) = 4 - 4 = 0$

* For limit, we have to split for left & right:

- $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} x^2 + 2x = (-2)^2 + 2\cdot(-2) = 0$ ($x < -2$)
- $\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} x^3 - 6x = (-2)^3 - 6\cdot(-2) = -8 + 12 = 4$ ($x > -2$)

$\rightarrow$ lim DNE $\Rightarrow$ Discont. is at $x=-2$

Ex 4. Cont'd : f is cont's everywhere BUT NOT at $x = -2$

$\Rightarrow$ Dom of Cont'y : $(-\infty, 2) \cup (2, +\infty)$

Ex 5. Is
$$f(x) = \begin{cases} \dfrac{x^3 - 27}{x^2 - 9} & x \neq 3 \\ \dfrac{9}{2} & x = 3 \end{cases}$$

(a) Cont's at $x = 3$?

Practice : (b) Cont's at $x = -3$?

(a) Check : $\lim_{x \to 3} f(x) = f(3)$.

- $f(3) = \dfrac{9}{2}$

To find the limit , we don't need to split into left/right limit, as we don't have $x > 3$ and $x < 3$ in the equations .

- $$\lim_{\boxed{x \to 3}} f(x) = \lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$$

$x \neq 3$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x^2 + 3x + 9)}{\cancel{(x-3)}(x+3)}$$

$$= \frac{3^2 + 3.3 + 9}{3 + 3} = \frac{27}{6} = \frac{9}{2}$$

$\Rightarrow \lim_{x \to 3} f(x) = f(3) \Rightarrow f$ is Cont's at $x = 3$

Practice 2. What is the domain of continuity of the following function?

$$f(x) = \begin{cases} \dfrac{x-6}{x-3} & x < 0 \\ 2 & x = 0 \\ \sqrt{4+x^2} & x > 0 \end{cases}$$

Question? Is there any rational function that is cont's everywhere? YES! Many ...

Rational functions for which their denominator is Never 0.

Examples. $y = \dfrac{x^7 + 5x^4 + 1}{x^2 + 1}$ $\rightsquigarrow x^2+1=0 \rightsquigarrow x^2 = -1$ NOT possible

or

$y = \dfrac{x}{x^6+8}$ $\rightsquigarrow x^6 = -8$ NOT possible

$\Longrightarrow$ Dom of cont'y : $(-\infty, +\infty)$

Ex 6. Find "b" such that $f(x)$ is cont's everywhere.

$$f(x) = \begin{cases} 2x^2 + b & x \geq -1 \\ -x^3 & x < -1 \end{cases}$$

(*). Here we already know the function is cont's, so we have, at the "border point": $\lim\limits_{x \to -1} f(x) = f(-1)$

$f(-1) = 2\cdot(-1)^2 + b = 2 + b$

Cont's fun: all must be equal

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x^2 + b = 2 + b$

$2 + b = 1$

$\Rightarrow \boxed{b = 1 - 2 = -1}$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -x^3 = -(\cancel{-1})^3 = +1$

-1

Practice 3. Find A & B such that

$$g(x) = \begin{cases} x^2 + Ax + B & x \le 1 \\ x - B & 1 < x \le 3 \\ x^3 + Ax & x > 3 \end{cases}$$

is cont's everywhere.

Some Rules.

$f(x)$ & $g(x)$ cont's at a point $x = a$

→ $f(x) + g(x)$

→ $f(x) - g(x)$

→ $f(x) \cdot g(x)$

⟹ cont's at $x = a$

→ $f(x)/g(x)$ → cont's where $g(x) \neq 0$

→ $f \circ g(x)$ → g cont's at a

f cont's at $g(a)$

⇝ $f \circ g$ cont's at a