Some Applied Problems : (Rate of Change)

Example 1. A ball is oscillating up and down, and its height (in feet) above the floor at time t is

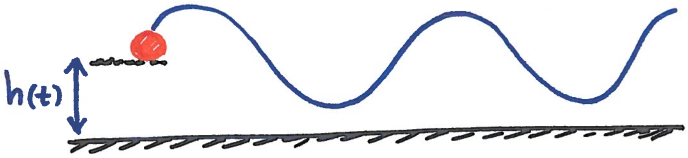
$$h(t) = 5 + 2\sin\left(\frac{t}{2}\right) \qquad (t \text{ is in radian.})$$

a) What is the height of ball after $\frac{\pi}{3}$ seconds ?

b) How fast is the balling moving after $\frac{\pi}{3}$ seconds ?

c) Is the ball moving up or down after $\frac{\pi}{3}$ seconds ?

d) Is the vertical velocity of the ball ever 0 ?



Key Point in solving applied problems is to correctly interprete the words as math concepts.

a) height $= h(t) \xrightarrow{t=\frac{\pi}{3}\,s} h\left(\frac{\pi}{3}\right) = 5 + 2\sin\left(\frac{\frac{\pi}{3}}{2}\right)$

$$= 5 + 2\underbrace{\sin\left(\frac{\pi}{6}\right)}_{\frac{1}{2}}$$

$$= 5 + 1 = 6 \text{ ft}$$

b) How fast $\rightsquigarrow$ speed or velocity $\rightsquigarrow$ derivative (rate of change) of height

We should then find $h'(\frac{\pi}{3})$.

First : $h'(t) = ?$

$$h(t) = 5 + 2\underbrace{\sin\left(\frac{t}{2}\right)}_{\text{Sin ▨}} \rightsquigarrow h'(t) = 0 + 2\underbrace{\cos\left(\frac{t}{2}\right)}_{\text{Derivative of Sin ▨ is Cos ▨}} \cdot \underbrace{\frac{1}{2}}_{\text{Derivative of inside : }\left(\frac{t}{2}\right)'}$$

Chain rule

$$\Rightarrow h'(t) = 2 \cdot \frac{1}{2} \cos\left(\frac{t}{2}\right) = \cos\left(\frac{t}{2}\right)$$

Now : $h'(\frac{\pi}{3}) = \cos\left(\frac{\frac{\pi}{3}}{2}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ ft/s

c) ball moving up/down $\rightsquigarrow$ height increasing / decreasing

$\rightsquigarrow$ $h'(t)$ positive / negative

$\xrightarrow{t=\frac{\pi}{3}}$ $h'(\frac{\pi}{3})$ $\oplus$ / $\ominus$

From part (b) : $h'(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ is positive

so the ball is moving up.

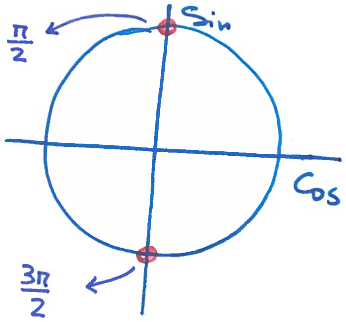
d) Velocity $= 0$ $\rightsquigarrow$ Derivative of height $= 0$

$\xrightarrow{\text{solve}}$ $h'(t) = 0$

From part (b) $h'(t) = \cos\left(\frac{t}{2}\right)$

So we must solve the equation :

$\cos\left(\frac{t}{2}\right) = 0$ ⟶ Where does Cos become 0 ?



when $\frac{t}{2} = \frac{\pi}{2} \Rightarrow \boxed{t = \pi \text{ s}}$

Or $\frac{t}{2} = \frac{3\pi}{2} \Rightarrow \boxed{t = 3\pi \text{ s}}$

In one cycle ; at $t = \pi_s$ and $t = 3\pi_s$ the vertical velocity becomes 0.

Other cycles : when $\frac{t}{2} = \frac{\pi}{2} + 2n\pi \xrightarrow[\text{by } 2]{\text{multiply}} t = \pi + 4n\pi$ s

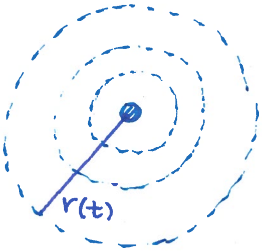
or $\frac{t}{2} = \frac{3\pi}{2} + 2n\pi \xrightarrow[\text{by } 2]{\text{multiply}} t = 3\pi + 4n\pi$ s

$n = 0, \pm 1, \pm 2, \ldots$

Example 2. A stone is dropped into a lake, creating a circular ripple whose radius at t seconds is

$$r(t) = (4t)^{\frac{1}{4}} \text{ cm}.$$

Find the rate of change of the area enclosed by the ripple at 3 seconds ?



1st : Formulate the area of ripples :

Area $\leftarrow A(r) = \pi r^2$

But r itself is a function of t, so:

$$A(r(t)) = \pi(r(t))^2 = \pi\left((4t)^{\frac{1}{4}}\right)^2 \underset{\text{Simplify}}{=} \pi(4t)^{\frac{1}{2}} = \pi\sqrt{4t} = 2\pi\sqrt{t}$$

$\Rightarrow$ Area as a function of time : $A(t) = 2\pi\sqrt{t}$

Now rate of change in area : $A'(t) \xrightarrow{t=3s} A'(3) = ?$

$$A'(t) = 2\pi\left(\frac{1}{2}t^{\frac{1}{2}-1}\right) = \pi t^{-\frac{1}{2}}$$

$\xrightarrow{\text{Substitute } t=3}$

make it a ⊕ power

$$A'(t) = \frac{\pi}{t^{\frac{1}{2}}} = \frac{\pi}{\sqrt{t}}$$

$$\Rightarrow \boxed{A'(3) = \frac{\pi}{\sqrt{3}}} \quad \text{cm}^2/\text{s}$$

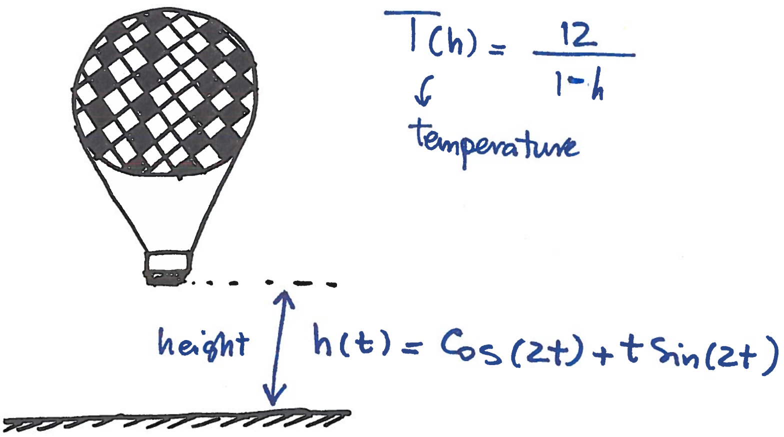
Example 3. You are riding in a balloon, and at time t (in minutes) you are $h(t) = \cos(2t) + t\sin(2t)$ feet high.

If the temperature at an elevation h is

$$T(h) = \frac{12}{1-h} \quad \text{degrees Celsius.}$$

a) How fast is the temperature changing?

b) Is the balloon temperature increasing or decreasing at $t = \pi/2$ minutes?



a) How fast : Rate of change with respect to time

1st $\rightsquigarrow$ Make temperature a function of time

$$T(h(t)) = \frac{12}{1-h(t)} = \frac{12}{1-(\cos(2t)+t\sin(2t))}$$

$\hookrightarrow$ a function of time now!

Rate of change ⇝ Derivative ⇝ $T'(t) = ?$

$$T(t) = \frac{\overbrace{12}^{f(t)}}{\underbrace{1 - \cos 2t - t \sin 2t}_{g(t)}}$$

Quotient Rule:

$$T'(t) = \frac{\overset{0}{f'}g - f g'}{g^2}$$

top

$f(t) = 12 \rightsquigarrow f'(t) = 0$

bottom

$$g(t) = 1 - \overbrace{\cos 2t}^{\cos ▨} - \overbrace{t \sin 2t}^{\text{product rule}}$$

sin ▨ — derivative of Sin ↑outside — $(2t)'$ ↑ inside

$$g'(t) = 0 - (\underbrace{-\sin(2t)}_{\text{outside}} \cdot \underbrace{2}_{\text{inside}}) - (1 \cdot \sin(2t) + t \cdot \cos(2t) \cdot 2)$$

$$= +2\sin(2t) - \sin 2t - 2t\cos(2t)$$

$$= \sin(2t) - 2t\cos(2t)$$

$$\Rightarrow T'(t) = \frac{-12(\sin(2t) - 2t\cos(2t))}{(1 - \cos 2t - t\sin 2t)^2}$$

b) Temperature increases/decreases $\xrightarrow{t=\frac{\pi}{2}}$ $T'(\frac{\pi}{2})$ ⊕ or ⊖ ?

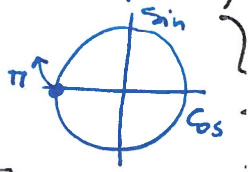
$$T'\left(\frac{\pi}{2}\right) = \frac{-12\left(\overset{0}{\cancel{\sin(\overset{\pi}{\cancel{2}\cdot\frac{\pi}{\cancel{2}}})}} - \cancel{2}\frac{\pi}{\cancel{2}}\overset{-1}{\cancel{\cos(\overset{\pi}{\cancel{2}\cdot\frac{\pi}{\cancel{2}}})}}\right)}{\left(1 - \overset{-1}{\cancel{\cos(\overset{\pi}{\cancel{2}\cdot\frac{\pi}{\cancel{2}}})}} - \frac{\pi}{2}\overset{0}{\cancel{\sin(\overset{\pi}{\cancel{2}\cdot\frac{\pi}{\cancel{2}}})}}\right)^2}$$

$$= \frac{-12(-\pi\cdot(-1))}{[1-(-1)]^2} = \frac{-12\pi}{4} = ⊖3\pi$$

↳ Temperature decreases.

$\sin \pi = 0$

$\cos \pi = -1$



Practice 1. If $P(x) = x^2 e^{-\frac{x}{100}}$ is the total value of the production when there are x workers in a factory, then the average productivity of the workforce is $A(x) = \frac{P(x)}{x}$.

How fast is the average productivity changing?

Practice 2. The percentage of a population, $P(t)$, who have heard a rumor by time t, is often modeled by

$$P(t) = \frac{100}{(1 + Ae^{-\frac{t}{2}})}$$

where A is a positive constant.

a) Initially what percentage of the population have heard the rumor? (In terms of A)

b) How fast is the rumor spreading?

Solution to Practices.

1) $$P(\overset{\#\text{ of workers.}}{x}) = x^2 e^{-\frac{x}{100}} \quad \text{total production}$$

$$\text{average production} \leftarrow A(x) = \frac{P(x)}{x} = \frac{\cancel{x^2} e^{-\frac{x}{100}}}{\cancel{x}} = x e^{-\frac{x}{100}}$$

Rate of change in $A(x) \rightsquigarrow A'(x) = ?$

Product Rule : $$A(x) = \underbrace{x}_{f}\underbrace{e^{-\frac{x}{100}}}_{g}$$

$$A'(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

$$= 1 \cdot e^{-\frac{x}{100}} + x \cdot \overbrace{e^{-\frac{x}{100}}}^{\text{outside}\,:\, e^{▨}} \cdot \overbrace{-\frac{1}{100}}^{\text{inside}\,:\,\left(-\frac{x}{100}\right)'}$$

$$= e^{-\frac{x}{100}} - \frac{x}{100} e^{-\frac{x}{100}} \underset{\text{simplify}}{=} \boxed{e^{-\frac{x}{100}}\left(1 - \frac{x}{100}\right)}$$

---

2) $$P(t) = \frac{\overset{f \rightsquigarrow f' = 0}{100}}{\underset{g}{(1 + Ae^{-\frac{t}{2}})}}$$

a) initially $\rightsquigarrow t = 0 \longrightarrow P(0) = \dfrac{100}{1 + Ae^0} = \dfrac{100}{1+A}$ % have heard the rumor.

b) How fast $\rightsquigarrow P'(t) = \dfrac{\overset{0}{\cancel{f'g}} - f g'}{g^2}$

$$g(t) = 1 + \underset{\text{constant}}{A}\, e^{-t/2} \rightarrow e^{▨}$$

$$g'(t) = 0 + A\, \overbrace{e^{-t/2}}^{\text{outside}} \cdot \overbrace{-\frac{1}{2}}^{\text{inside}}$$

$$P'(t) = \frac{-100 \cdot A e^{-t/2} \cdot -1/2}{(1 + Ae^{-t/2})^2} = \frac{50 A e^{-t/2}}{(1 + Ae^{-t/2})}$$

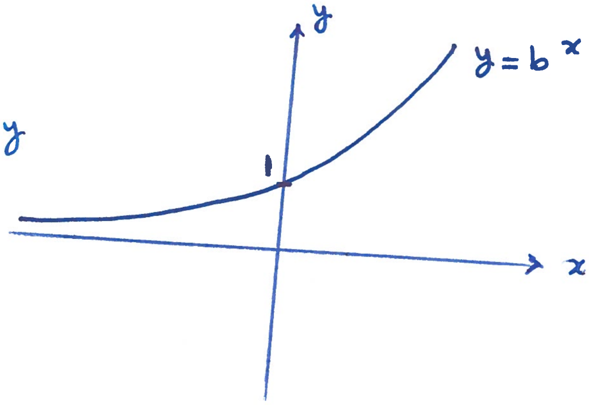
logarithmic functions:

The logarithmic functions are closely related to the exponential functions, so we briefly recall the function:

Review: $y = b^x$ where $b \neq 0, b \neq 1, b > 0$

Graph:

Function starts very small and then it grows very fast. (exponentially fast)



Properties:

- y-int: $y=1$ ⇝ $(0,1)$ is ALWAYS on the graph.
- x-int: NONE ⇝ The graph NEVER touches the x-axis.
- Domain: $(-\infty, \infty)$ ⇝ All real numbers are OK to be plugged in for x.
- Range: $(0, +\infty)$ ⇝ NEVER goes below x-axis
  ⇝ ALWAYS Positive.

Natural exponential function:

When the base "b" is the number "e"

then $y = e^x$ is called the natural exp function.

Definition. $y = \log_b x$ is called the logarithmic function and it is defined as the inverse function of $y = b^x$.

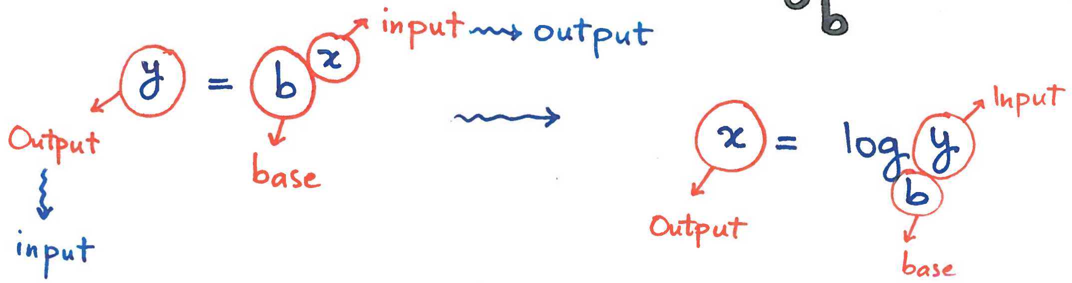
Recall: $y = \sqrt{x}$ and $y = x^2$ are inverse of each other.

$y = \sqrt[3]{x}$ and $y = x^3$ are inverse of each other

Now $y = \log_b x$ and $y = b^x$ are inverse of each other.

Remark. We don't need to worry if $y = b^x$ is invertible. Why? Because as you see the graph of $y = b^x$ passes the horizontal line test, thus it is 1-1 and so invertible.

⤳ Go from $y = b^x$ to $y = \log_b x$ and vice verca.



But by convention, we'd rather denote input by x and output by y:

$$y = \log_b x$$

Example.

$$2^{4} = 16 \rightsquigarrow \log_{2} 16 = 4$$

result

$$\log_3 9 = \overset{?}{\boxed{}} \xrightarrow[\text{into exp}]{\text{translate}} 3^{\boxed{2}} = 9 = 3^{2}$$

raise

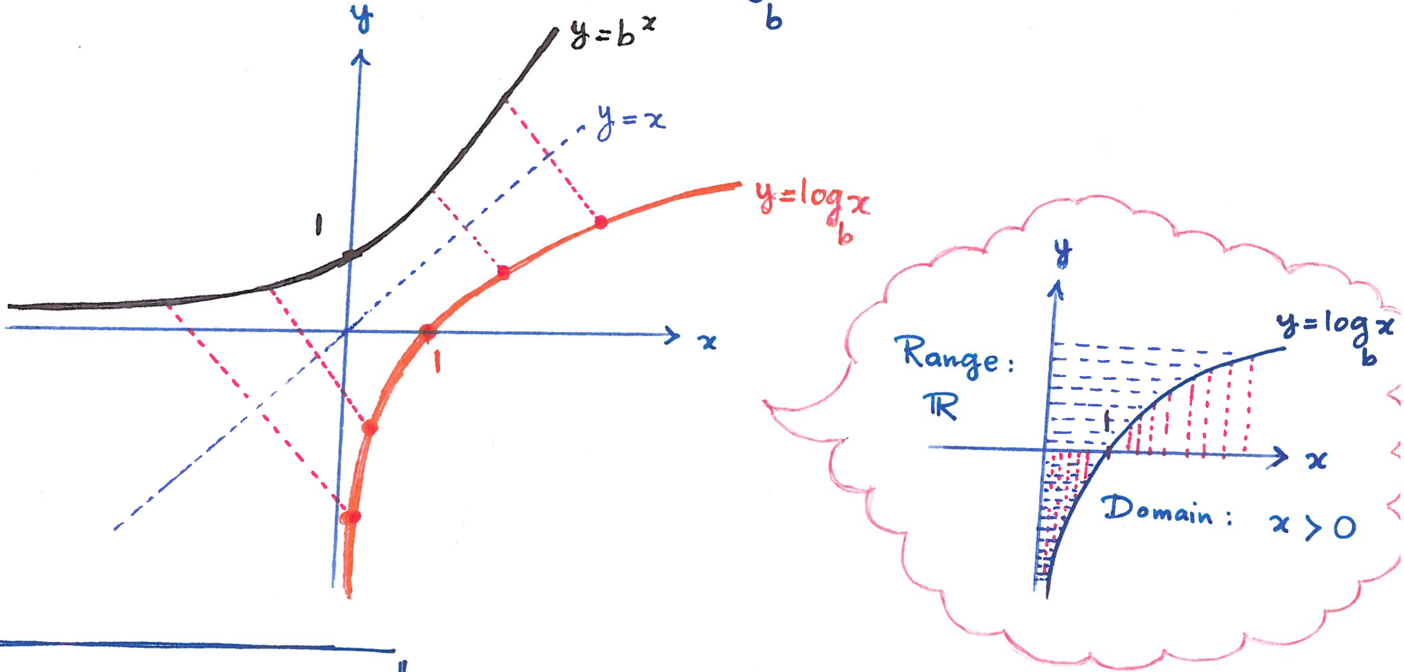
Graphical interpretation of $y = \log_b x$

We have seen in inverse functions that the two functions

$$y = f(x) \quad \text{and its inverse} \quad y = f^{-1}(x)$$

are reflection of each other over the line $y = x$.

We know the graph of $y = b^x$, we reflect its graph over $y = x$ to find the graph of $y = \log_b x$.



Properties of $y = \log_b x$

intercepts and domain and range of $y = b^x$ must be interchanged.

- x-int : $x = 1 \rightsquigarrow \log_b 1 = 0$
  $\rightsquigarrow (1, 0)$ ALWAYS on the graph
- y-int : NONE $\rightsquigarrow$ the graph NEVER touches the y-axis.
- Domain : $(0, +\infty) \rightsquigarrow$ 0 and negative numbers are NOT OK and can NOT be plugged in for x.
- Range : $(-\infty, \infty) \rightsquigarrow$ function produces all real numbers.

Useful Remark. Looking closely at the graph of $y = \log_b x$, we can notice that when $x > 1$, the log function is positive (above x-axis), and for $0 < x < 1$, log becomes negative (below x-axis.)

In other words,

$$x > 1 \Longleftrightarrow \log_b x > 0$$

$$0 < x < 1 \Longleftrightarrow \log_b x < 0$$

$$x = 1 \Longleftrightarrow \log_b 1 = 0$$

Example.

$$\log_2 \overset{>1}{\boxed{1.5}} > 0 \quad , \quad \log_3 \overset{<1}{\boxed{\tfrac{1}{2}}} < 0 \quad , \quad \log_{10} \overset{<1}{\boxed{\tfrac{1}{9}}} < 0$$

positive

Natural Logarithm

As we saw earlier, the case when the base $b = e$ is special.

For log function, when $b = e$: $y = \log_b x = \log_e x$

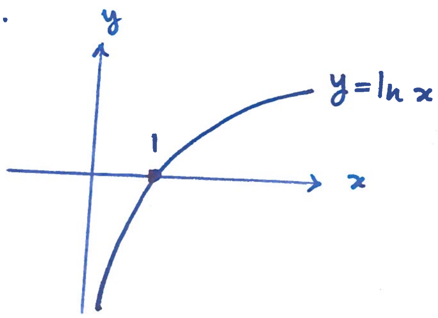
We'd rather remove the base and chang log to ln.

Therefore :

$$y = \log_e x \xrightarrow[\text{Notation}]{} \boxed{y = \ln x}$$

All the properties of log also hold true for ln :

Graph.



- Domain : $(0, \infty)$
- Range : $(-\infty, +\infty)$
- $\ln 1 = 0 \rightsquigarrow$ x-int : $x = 1$
- NO y-int

Example.

$\ln e = \boxed{1}$

base: e
Hidden

$\rightsquigarrow e^{\square} = e$

$\ln e^2 = \boxed{2} \rightsquigarrow e^{\square} = e^2$

e
Hidden

Law of Logarithms:

I • $\log_b b = 1$ in particular : $\boxed{\ln e = 1}$

II • $\log_b 1 = 0$ in particular : $\boxed{\ln 1 = 0}$

multiplication breaks into addition

III • $\boxed{\log_b AB = \log_b A + \log_b B}$

in particular $\boxed{\ln AB = \ln A + \ln B}$

Wrong cases.

$\xcancel{\ln A+B = \ln A + \ln B}$

$\xcancel{\ln A+B = \ln A \cdot \ln B}$

$\xcancel{\ln A \cdot B = \ln A \cdot \ln B}$

division breaks into subtraction.

IV • $\boxed{\log_b \frac{A}{B} = \log_b A - \log_b B}$

in particular : $\boxed{\ln A/_B = \ln A - \ln B}$

Wrong cases :

$$\xcancel{\ln A-B = \ln A - \ln B}$$

$$\xcancel{\ln A - B = \ln A / \ln B}$$

$$\xcancel{\ln A/_B = \frac{\ln A}{\ln B}}$$

V • $\log_b A^{r} = r \log_b A$

in particular : $\boxed{\ln A^r = r \ln A}$

Examples .

- $\log_{10} 5 + \log_{10} 2 \overset{III}{=} \log_{10} 5 \cdot 2 = \log_{10} 10 \overset{I}{=} 1$
- $\log_3 \frac{1}{9} \overset{IV}{=} \log_3 1 - \log_3 9 \overset{II}{=} 0 - \log_3 9 = 3^2$

  $= -\log_3 3^{\textcircled{2}}$

  $\overset{V}{=} -2 \log_3 3 \overset{I}{=} -2$
- $\ln \sqrt[3]{e} = \ln e^{\frac{1}{3}} \overset{V}{=} \frac{1}{3} \ln e \overset{I}{=} \frac{1}{3}$
- $\ln e^2 - \ln \sqrt[5]{e^2} \overset{IV}{=} \ln \frac{e^2}{\sqrt[5]{e^2}} = \ln \frac{e^2}{e^{2/5}} = \ln e^{2-\frac{2}{5}} = \ln e^{\textcircled{\frac{8}{5}}}$

  $= \frac{8}{} \cancelto{1}{\ln e} = \frac{8}{}$