Homework 5, MATH 110-003

Due date: Tuesday, Nov 22, 2016 (in class)

Hand in full solutions to the questions below. Make sure you justify all your work and include complete arguments and explanations. Your answers must be clear and neatly written, as well as legible (no tiny drawings or micro-handwriting please!). ***Your answers must be stapled, with your name and student number at the top of each page.***

1. Two cars start moving from the same point in two directions that makes 90 degrees at the speed of 3 m/s, and 4 m/s. (Recall: speed=$\dfrac{\text{distance}}{\text{time}}$)

   a) What is the distance between the two cars after 2 seconds?

   b) How fast is the distance between the two cars changing as a function of t?

   c) How fast is the distance between the two cars changing at 2 seconds?(Include the units)

2. A spherical balloon is being inflated such that its radius r at t seconds is $r(t) = e^{\frac{3t}{t+1}}$ cm. Find the rate of increase of the surface area of the balloon.
   (Recall: Surface area: $S = 4\pi r^2$)

3. Suppose $f(x) = \frac{1}{\pi} x \tan x + \ln(x^2 + 1)$,

   a) What is the domain of f?

   b) Is f increasing or decreasing at $x = \pi$? (Show complete work.)

   Removed

   ~~c) Show that there is a point x in $[0, \pi]$ such that the graph of f crosses the line $y = 1$. (Justify your~~ answer.)

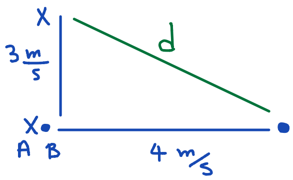
4. This problem deals with functions called the *hyperbolic sine* and *hyperbolic cosine*. These functions occur in electromagnetic theory, heat transfer, fluid dynamics and etc. Hyperbolic sine and cosine are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

   a) Find derivatives of $\sinh x$ and $\cosh x$, and express your answer in terms of $\sinh x$ and $\cosh x$.

   b) Use part (a) to find derivatives of $y = x \sinh(\dfrac{1}{x})$ and $y = \cosh(x^2)$.

(1)



(a)

Car A travels 3 meters at each second $\xrightarrow{2\text{ seconds}}$ Car A : 6 m

Car B 〃 4 meters 〃 〃 〃 $\longrightarrow$ Car B : 8 m

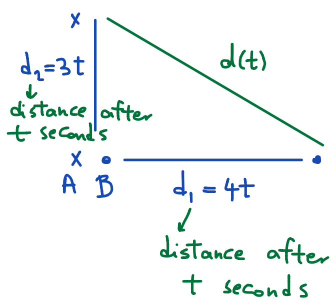
Pythagoras $\Rightarrow$ $d = \sqrt{6^2 + 8^2} = 10$ m $\rightsquigarrow$ the distance between two cars after 2 seconds.

(b) How fast $\rightsquigarrow$ Rate of Change in distance between two cars $\rightsquigarrow$ $d'(t)$

1st step is to write the distance as a function of t :

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \xrightarrow{\text{Car A}} \quad 3 = \frac{d_1}{t} \Rightarrow d_1 = 3t$$

$$\xrightarrow{\text{Car B}} \quad 4 = \frac{d_2}{t} \Rightarrow d_2 = 4t$$



$$\Rightarrow d(t) = \sqrt{(3t)^2 + (4t)^2} = \sqrt{9t^2 + 16t^2} = \sqrt{25t^2} = 5t$$

$\Rightarrow$ distance : $d(t) = 5t$

$\Rightarrow$ speed : $\boxed{v(t) = d'(t) = 5}$ m/s

(c) By part (b) speed is always constant and equal to 5 m/s.

So at t=2 the distance is changing 5 m/s.

---

2) $r(t) = e^{\frac{3t}{t+1}}$

$$S(r) = 4\pi r^2 \xrightarrow[\text{of } t]{\text{make it a function}} S(t) = 4\pi\left(e^{\frac{3t}{t+1}}\right)^2$$

Rate of change in the area $\rightsquigarrow$ $S'(t) = \underbrace{4\pi}_{\text{Constant remains}} \underbrace{\left(2\, e^{\frac{3t}{t+1}}\right)}_{\text{Derivative of outside : } \bullet^2} \underbrace{\left(e^{\frac{3t}{t+1}}\right)}_{\text{Derivative of middle : } e^{\bullet}} \underbrace{\frac{\overset{3t+3-3t}{3(t+1) - 3t}}{(t+1)^2}}_{\text{Derivative of inside : Quotient Rule}}$

$$\Rightarrow \quad S'(t) = 24\pi \frac{\left(e^{\frac{3t}{t+1}}\right)^2}{(t+1)^2} = 24\pi \frac{e^{\frac{6t}{t+1}}}{(t+1)^2}$$

3) $f(x) = \frac{1}{\pi} x \tan x + \ln(x^2+1)$

a) two functions with restriction on x :

- $\tan x = \frac{\sin x}{\cos x} \Rightarrow$ Dom = Everywhere except where $\cos x = 0$

$$\cos x = 0 \xrightarrow[\text{cycle}]{\text{One}} x = \frac{\pi}{2}, \ x = \frac{3\pi}{2}$$

$$\xrightarrow[\text{cycles}]{\text{other}} x = \frac{\pi}{2} + 2n\pi, \ x = \frac{3\pi}{2} + 2n\pi \qquad n = 0, \pm 1, \pm 2, \dots$$

$$\Rightarrow \text{Dom} = \text{Everywhere except } \left\{ x = \frac{\pi}{2} + 2n\pi, \ x = \frac{3\pi}{2} + 2n\pi \right\}$$
$$n = 0, \pm 1, \pm 2, \dots$$

- $\ln(x^2+1)$ is defined when $x^2+1 > 0$

But x^2+1 is always positive so it is always true to have $x^2+1 > 0$, NO x to exclude.

No change in the domain comes from $\ln(x^2+1)$

$$\Rightarrow$$

$$\text{Dom} = \text{Everywhere except } \left\{ x = \frac{\pi}{2} + 2n\pi, \ x = \frac{3\pi}{2} + 2n\pi \right\}$$
$$n = 0, \pm 1, \pm 2, \dots$$

b) increasing or decreasing $\rightsquigarrow f'(x)$

$$f'(x) = \frac{1}{\pi}\left(1 \cdot \tan x + x(1+\tan^2 x)\right) + \frac{2x}{x^2+1}$$

$$f'(\pi) = \frac{1}{\pi}\left(\underbrace{\tan \pi}_{0} + \pi(1+\underbrace{\tan^2 \pi}_{0})\right) + \frac{2\pi}{\pi^2+1} = \frac{1}{\pi} \cdot \pi + \frac{2\pi}{\pi^2+1} = 1 + \frac{2\pi}{\pi^2+1} > 0$$

$\Rightarrow f$ is increasing at $x = \pi$.

$$\sinh x = \frac{e^{x} - e^{-x}}{2}, \quad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\begin{cases} (\sinh x)' = \dfrac{e^{x} - e^{-x}\cdot(-1)}{2} = \dfrac{e^{x} + e^{-x}}{2} = \cosh x \\[2ex] (\cosh x)' = \dfrac{e^{x} + e^{-x}\cdot(-1)}{2} = \dfrac{e^{x} - e^{-x}}{2} = \sinh x \end{cases}$$

→ We don't need to use quotient rule as the denominator is the constant $\frac{1}{2}$.
But if you do use the rule, you'll get the same answer.

Conclusion:

$$y = \sinh x \longrightarrow y' = \cosh x$$

$$y = \cosh x \longrightarrow y' = \sinh x$$

(b) $y = \underbrace{x}_{f}\ \underbrace{\sinh\left(\frac{1}{x}\right)}_{g}$

product rule: $y' = \overset{1}{f'}\,g + f\,g' = \sinh\left(\frac{1}{x}\right) + x \cdot \frac{-1}{x^2}\cosh\left(\frac{1}{x}\right)$

$g(x) = \sinh\left(\frac{1}{x}\right) = \sinh$

$= \sinh\left(\frac{1}{x}\right) - \frac{1}{x}\cosh\left(\frac{1}{x}\right)$

$g'(x) = \underbrace{\cosh\frac{1}{x}}_{\text{outside derivative from part (a)}} \cdot \underbrace{\frac{-1}{x^2}}_{\text{inside derivative: quotient rule}}$

$y = \underbrace{\cosh}_{\text{outside}}\ \underbrace{(x^2)}_{\text{inside}} \Rightarrow y' = \sinh(x^2)\cdot 2x$