Homework 2, MATH 110-003

Due date: Tuesday, Oct 11, 2016 (in class)

Hand in full solutions to the questions below. Make sure you justify all your work and include complete arguments and explanations. Your answers must be clear and neatly written, as well as legible (no tiny drawings or micro-handwriting please!). ***Your answers must be stapled, with your name and student number at the top of each page.***

1. The temperature for the month of May can be modelled by the equation

$$T(t) = 54(\frac{t}{2t+3})$$

where T is the temperature in Celsius (C), and t is the day of month.
Use the limit formula to find the instantaneous rate of change in temperature on May 3^{rd}.

2. Find the points at which the following functions are **discontinuous**?

(a) $f(x) = 3$, (b) $g(x) = 1 + |x-1|$, (c) $h(x) = \dfrac{x+1}{x^2-1}$,
(d) $F(t) = \dfrac{2t}{5+t^2}$, (e) $G(u) = \dfrac{u+4}{u+4}$

3. Determine the values of the constants A and B so that the following function is continuous for all values of x.

$$g(x) = \begin{cases} Ax - B, & x \le -1 \\ 2x^2 + 3Ax + B, & -1 < x \le 1 \\ 4, & x > 1 \end{cases}$$

4. Prove that the function

$$p(x) = x^3 - 15x + 1$$

has three roots in the interval $[-4, 4]$. Make sure to state any assumptions you are making, or any theorems you are using.

1) $T(t) = 54\left(\frac{t}{2t+3}\right)$

May 3rd

Instantaneous Rate of Change at $\underbrace{t=3}$: $\lim_{h \to 0} \frac{T(3+h) - T(3)}{h}$

- Recall. Inst. Rate of Change is the slope of the tangent line

Start with secant line in a small interval around 3 : $[3, 3+h]$ and make h smaller $(h \to 0)$ so that secant line approaches tangent line and we get the limit formula.

$T(t)$ is given, we can find $T(3)$ and $T(3+h)$:

$$T(3) = 54\left(\frac{3}{2 \cdot 3 + 3}\right) = 54\left(\frac{3}{9}\right) = 54\left(\frac{1}{3}\right)$$

$$T(3+h) = 54\left(\frac{3+h}{\underbrace{2 \cdot (3+h) + 3}_{6+2h}}\right) = 54\left(\frac{3+h}{9+2h}\right)$$

Plug them into the limit formula:

$$\lim_{h \to 0} \frac{54\left(\frac{3+h}{9+2h}\right) - 54\left(\frac{1}{3}\right)}{h} = \lim_{h \to 0} \frac{54\left[\frac{(3+h) \cdot 3}{9+2h} - \frac{1 \cdot (9+2h)}{3}\right]}{h}$$

$$= \lim_{h \to 0} \frac{54\left[\frac{9+3h-(9+2h)}{3(9+2h)}\right]}{h}$$

$$= \lim_{h \to 0} \frac{54 \cdot \frac{h}{27+6h}}{h}$$

$$= \lim_{h \to 0} \frac{54\cancel{h}}{\cancel{h}(27+6h)}$$

Now we have the formula simplified, after cancellation of h, we get rid of $0/0$ and now we have:

$$\lim_{h \to 0} \frac{54}{27+6h} = \boxed{\frac{54}{27} = 2} \to \text{instantaneous rate of change}$$

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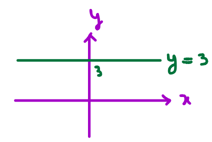
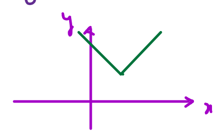
2) (a) $f(x) = 3 \rightsquigarrow$ Constant function $\rightsquigarrow$ Continuous everywhere

NO where Discont's !



(b) $g(x) = 1 + |x-1| \rightsquigarrow$ Absolute-value function $\rightsquigarrow$ Cont's Everywhere

No point of discontinuity !



(c) $h(x) = \dfrac{x+1}{x^2-1}$ $\qquad \dfrac{x+1}{(x+1)(x-1)} \rightsquigarrow$ Cancellation does NOT count for points of discont'y.

Points of discont'y : denom = 0 $\Rightarrow x^2 - 1 = (x-1)(x+1) = 0$

$\Rightarrow \boxed{x = 1 \ , \ x = -1}$ points where h is Discont's

(d) $F(t) = \dfrac{2t}{5+t^2}$

$5 + t^2 = 0 \Rightarrow t^2 = -5 \to$ NO point of discont'y

(e) $G(u) = \dfrac{u+4}{u+4}$ Even though top and bottom are cancelled, still the function is NOT defined at $u = -4$.

$\Rightarrow$ Point of discont'y $= -4$

3) $$g(x) = \begin{cases} Ax - B & x \le -1 \\ 2x^2 + 3Ax + B & -1 < x \le 1 \\ 4 & x > 1 \end{cases}$$

To have the function cont's everywhere, it must be cont's at $x=-1$ and $x=1$. Check left & right limit for each and equate:

At $x=-1$: $f(-1) = A\cdot(-1) - B = -A - B$

f cont's at $x=-1$

$$\lim_{x\to -1^-} f(x) = A\cdot(-1) - B = -A - B$$

$$\lim_{x\to -1^+} f(x) = 2\cdot(-1)^2 + 3A\cdot(-1) + B = 2 - 3A + B$$

$$-A - B = 2 - 3A + B$$

$$2A - 2B = 2$$

$$\boxed{A - B = 1} \quad \mathrm{I}$$

At $x=1$: $f(1) = 2\cdot(1)^2 + 3A\cdot 1 + B = 2 + 3A + B$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2 + 3Ax + B = 2 + 3A + B$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4 = 4$$

f cont's at $x=-1$

$$2 + 3A + B = 4$$

$$\boxed{3A + B = 2} \quad \mathrm{II}$$

From I and II, we should solve

$$\begin{cases} A - B = 1 \Rightarrow B = A - 1 \\ 3A + B = 2 \end{cases}$$

$$3A + (A-1) = 2 \Rightarrow 3A = 3 \Rightarrow \boxed{A = 1} \Rightarrow \boxed{B = A - 1 = 0}$$

4) $f(x) = x^3 - 15x + 1$ over $[-4, 4]$

$f(x)$ is a polynomial, and it is Cont's in $[-4, 4]$

$$f(-4) = (-4)^3 - 15 \cdot (-4) + 1 = -64 + 60 + 1 = -3 \quad -$$
$$f(4) = (4)^3 - 15 \cdot 4 + 1 = 64 - 60 + 1 = 5 \quad +$$

By IVT, there is a root in $[-4, 4]$

to find 3 roots:

split into smaller intervals:

Root Root Root

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-3	19	23	15	1	-13	-21	-17	5

There are 3 roots each in the intervals: $[-4, -3]$, $[0, 1]$, $[3, 4]$