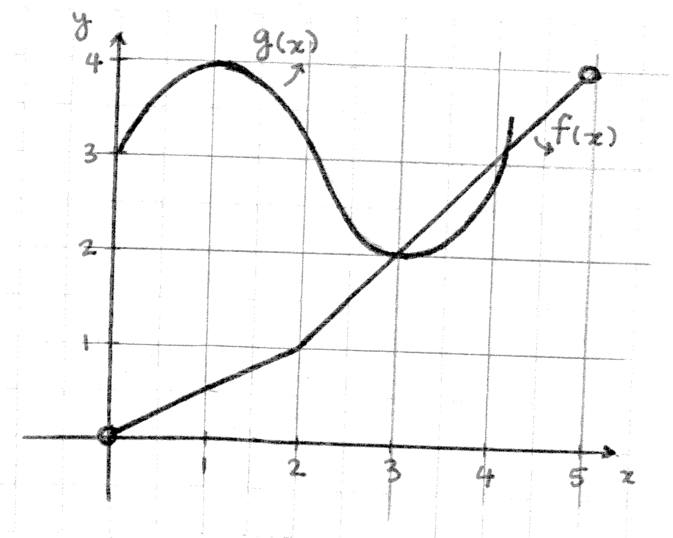
Homework 4, MATH 110-003

Due date: Tuesday, Nov 8, 2016 (in class)

Hand in full solutions to the questions below. Make sure you justify all your work and include complete arguments and explanations. Your answers must be clear and neatly written, as well as legible (no tiny drawings or micro-handwriting please!). ***Your answers must be stapled, with your name and student number at the top of each page.***

Question 1. The functions f and g are given by the following graphs.



Find the following values:

a) Domain of f'

b) Roots of g'

c) $(fg)'(3)$

d) $\left(\frac{f}{g}\right)'(1)$

Question 2. On what intervals is the function $f(x) = x^3 e^x$ increasing?

Question 3. Suppose

$$f(t) = \begin{cases} e^t \sin t & t < k \\ e^t \cos t & t > k \end{cases}$$

a) For what values of k does $\lim_{t \to k} f(t)$ exist?

b) Show that $f(t)$ is NOT continuous at the smallest k you found in part (a)?

c) Add a line to the function f such that it becomes continuous at the smallest k you found in part (a), and call this new function $g(t)$?

d) Is $g(t)$ differentiable at at the smallest k you found in part (a)?

Question 4. The function $f(x)$ is given by

$$f(x) = \frac{2\sin x}{e^x(1+\cos x)}$$

a) Find the domain of f. (Interval notation is NOT required.)

b) Find the derivative of f.

c) Find the equation of the tangent line to $f(x)$ at $x = 0$.

Question 5. Use product rule to prove that

$$\frac{d}{dx}(f(x))^3 = 3(f(x))f'(x)$$

(Hint: By using product rule first show that $(fgh)' = f'gh + fg'h + fgh'$, and then take $f = g = h$.)
(Recall that $\frac{d}{dx}$ is another notation for the derivative.)

1) a) f' is NOT defined when the graph of f has a corner.
At $x=2$, f has two distinct tangent line so $x=2$ is NOT in the domain of f'. Also endpoints must be excluded

$\rightsquigarrow$ **Domain of $f' = (0,2) \cup (2,5)$**

b) Roots of $g' \rightsquigarrow$ Points where $g'=0 \rightsquigarrow$ Horizontal tangent line to g

$\rightsquigarrow x=1 \,,\, x=3$

$\Rightarrow$ **Roots of g': $x=1$ & $x=3$**

c) $(fg)'(3) = f'(3)\,g(3) + f(3)\,g'(3) = 1 \cdot 2 + 2 \cdot 0 = 2$

$f'(3) =$ slope at $x=3$: $\frac{\text{Rise}}{\text{Run}} = \frac{3-2}{4-3} = 1$

$g(3) = 2$

$f(3) = 2$

$g'(3) = 0 \rightsquigarrow$ horizontal tangent line at $x=3$

d) $\left(\frac{f}{g}\right)'(1) = \frac{f'(1)\,g(1) - f(1)\,g'(1)}{(g(1))^2} = \frac{\frac{1}{2}\cdot 4 - f(1)\cdot 0}{4^2} = \frac{2}{16} = \frac{1}{8}$

$f'(1) =$ slope $= \frac{1-0}{2-0} = \frac{1}{2}$

$g(1) = 4$

$g'(1) = 0 \rightsquigarrow$ horizontal tangent line at $x=1$

$f(1) = \frac{1}{2}$ (NOT required really!)

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2) function f increasing $\rightsquigarrow$ derivative f' non-negative $f' \geq 0$

$f(x) = x^3 e^x \xrightarrow[\text{Rule}]{\text{Product}} f'(x) = (3x^2)(e^x) + (x^3)e^x$

$= \underline{x^2}\,\underline{e^x}\,\underline{(3+x)}$

We must have $f'(x) \geq 0$; f' consists of multiplication of three terms, so each of these three terms must be non-negative

x^2 and e^x are always non-negative, so the only term to check is
$3+x$: We must have $3+x \geq 0 \Rightarrow x \geq -3$

$\Rightarrow$ Interval : $[-3, +\infty)$

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3) a) To have $\lim_{t \to k} f(t)$ exist we must have

$$\lim_{t \to k^-} f(t) = \lim_{t \to k^+} f(t)$$

$$\lim_{t \to k^+} e^t \sin t = e^k \sin k \quad (\mathrm{I})$$

$$\lim_{t \to k^-} e^t \cos t = e^k \cos k \quad (\mathrm{II})$$

$\mathrm{I} = \mathrm{II}$

$\Longrightarrow \cancel{e^k} \cos k = \cancel{e^k} \sin k \rightarrow \frac{\sin k}{\cos k} = \frac{\cos k}{\cos k} \xrightarrow[\text{by } \cos k]{\text{divide}} \tan k = 1$

We should solve the trig equation : $\tan k = 1$

what angle has its $\tan = 1 \leadsto k = \frac{\pi}{4}$

tan repeats itself every π

$\xrightarrow[\text{angles}]{\text{other}} k = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, \ldots$

$= \frac{\pi}{4} + n\pi \qquad n = 0, \pm 1, \pm 2, \ldots$

b) smallest $k = \frac{\pi}{4}$

$$f(t) = \begin{cases} e^t \sin t & t < \frac{\pi}{4} \\ e^t \cos t & t > \frac{\pi}{4} \end{cases} \Longrightarrow \begin{array}{l} \lim_{t \to \pi/4^-} f(t) = e^{\frac{\pi}{4}} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} \\ \lim_{t \to \pi/4^+} f(t) = e^{\frac{\pi}{4}} \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} \end{array}$$

$f(\frac{\pi}{4})$ = NOT defined

$\rightarrow$ f NOT cont's at $t = \frac{\pi}{4}$, although the limit exists at $\frac{\pi}{4}$ but function is NOT defined at $\frac{\pi}{4}$.

c) To make f cont's we assume $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}$, so that:

$$\lim_{t \to \frac{\pi}{4}^-} f(t) = \lim_{t \to \frac{\pi}{4}^+} f(t) = f\left(\frac{\pi}{4}\right)$$

$\Rightarrow$ New function:

$$g(t) = \begin{cases} e^t \sin t & t < \frac{\pi}{4} \\ \frac{\sqrt{2}}{2} e^{\pi/4} & t = \frac{\pi}{4} \\ e^t \cos t & t > \frac{\pi}{4} \end{cases}$$

d) To check the differentiability of $g(t)$ at $t = \frac{\pi}{4}$, we must check if the derivative of g at left and right of $\frac{\pi}{4}$ are equal or Not.

$t < \frac{\pi}{4} \rightsquigarrow g(t) = e^t \sin t \rightsquigarrow g'(t) = e^t \sin t + e^t \cos t$

$\rightsquigarrow g'\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}} \sin\frac{\pi}{4} + e^{\frac{\pi}{4}} \cos\frac{\pi}{4} = e^{\frac{\pi}{4}} \frac{\sqrt{2}}{2} + e^{\frac{\pi}{4}} \frac{\sqrt{2}}{2}$

$= \sqrt{2}\, e^{\pi/4}$

$t > \frac{\pi}{4} \rightsquigarrow g(t) = e^t \cos t \rightsquigarrow g'(t) = e^t \cos t + e^t(-\sin t)$

$\rightsquigarrow g'\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}} \cos\frac{\pi}{4} - e^{\frac{\pi}{4}} \sin\frac{\pi}{4} = e^{\frac{\pi}{4}} \frac{\sqrt{2}}{2} - e^{\frac{\pi}{4}} \frac{\sqrt{2}}{2}$

$= 0$

$g'\left(\frac{\pi}{4}\right)$ has two different values at left & right of $\frac{\pi}{4}$

$\rightsquigarrow$ two different slopes at $\frac{\pi}{4}$ $\rightsquigarrow$ NOT differentiable at $\frac{\pi}{4}$.

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4) $f(x) = \dfrac{2 \sin x}{e^x (1 + \cos x)}$

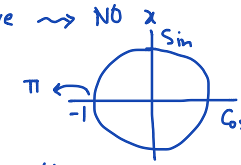
a) Domain of f = everywhere except the x's that $e^x(1 + \cos x) = 0$

$e^x(1 + \cos x) = 0 \rightsquigarrow e^x = 0 \rightsquigarrow e^x$ ALWAYS positive $\rightsquigarrow$ NO x

$1 + \cos x = 0 \rightsquigarrow \cos x = -1$

$\rightsquigarrow x = \pi, 3\pi, 5\pi$

$\rightarrow x = (2n+1)\pi \quad n = 0, \pm 1, \cdots$



$\Rightarrow$ Domain = everywhere except $\{x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots\}$

b) $f(x) = \dfrac{\overbrace{2\sin x}^{g}}{\underbrace{e^x(1+\cos x)}_{h}}$

Quotient Rule $\rightsquigarrow$ $f'(x) = \dfrac{g'(x)\,h(x) - g(x)\,h'(x)}{(h(x))^2}$

$g'(x) = 2\cos x$

$h'(x) = e^x(1+\cos x) + (-\sin x)\,e^x$ (product rule)

$= e^x + e^x\cos x - e^x \sin x$

$\Rightarrow f'(x) = \dfrac{\overbrace{2\cos x}^{g'} \cdot \overbrace{e^x(1+\cos x)}^{h} - \overbrace{(e^x + e^x\cos x - e^x\sin x)}^{h'} \cdot \overbrace{2\sin x}^{g}}{\left(e^x(1+\cos x)\right)^2}$

$2e^x(\cos^2 x + \sin^2 x) = 2e^x \cdot 1$

$= \dfrac{2e^x\cos x + 2e^x\cos^2 x - 2e^x\sin x - 2e^x\cos x\sin x + 2e^x\sin^2 x}{e^{2x}(1+\cos x)^2}$

$= \dfrac{2e^x(\cos x - \sin x - \sin x\cos x + 1)}{e^{2x}(1+\cos x)^2}$

c) at $x = 0 \rightsquigarrow m_{tan} = f'(0) = \dfrac{2e^0(\cos 0 - \sin 0 - \sin 0 \cos 0 + 1)}{e^0(1+\cos 0)^2}$

$= \dfrac{2 \cdot 2}{2^2} = 1$

$x = 0 \rightsquigarrow y = \dfrac{2\sin 0}{e^0(1+\cos 0)} = 0$

Point $(0,0)$ $\Longrightarrow$ $y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$ $\Rightarrow$ $\boxed{y = x}$

5) Prove $\frac{d}{dx}(f(x))^3 = 3(f(x))^2 f'(x)$

$$(\underbrace{f}_{\text{I}}\underbrace{g\,h}_{\text{II}})' = \underbrace{f'}_{\text{I}'}\underbrace{(gh)}_{\text{II}} + \underbrace{f}_{\text{I}}\underbrace{(gh)'}_{\text{II}'}$$

→ Again Product Rule

↓ Product Rule

$$= f'.gh + f.[g'h + gh']$$

$$= f'.gh + fg'h + fgh'$$

Now take $f = g = h$

$$(fgh)' = (f.f.f)' = (f^3)' = f'.(ff) + f(f'f) + fff'$$

$$= f'f^2 + f'f^2 + f'f^2$$

$$= 3f'(x)(f(x))^2 \quad \checkmark$$