Second Derivative

Dec 1
dec 25 ⇝ Last Lecture ☹

For a function $f(x)$, we know that by using different rules that we've learned, we can find its derivative $f'(x)$. Now we can apply the same rules on f' to find the derivative of the derivative : $(f')'$ and we denote this 2nd derivative $f''(x)$, or sometimes $\frac{d^2f}{dx^2}$ or $\frac{d^2y}{dx^2}$

Example. a) $f(x) = x^5 + 3x^7 + \sqrt{x}$

$$f(x) = x^5 + 3x^7 + x^{\frac{1}{2}}$$

$$f'(x) = \underline{5\,x^4} + \underline{21}\,\underline{x^6} + \underline{\frac{1}{2}}\,x^{\underline{\frac{1}{2}-1=-\frac{1}{2}}}$$

$$f''(x) = 5(4x^3) + 21(6x^5) + \frac{1}{2}\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= 20x^3 + 126x^5 - \frac{1}{4}x^{-\frac{3}{2}}$$

b) $g(x) = \underbrace{x^2 \underbrace{\sin^2(x)}_{\text{chain rule}}}_{\text{product rule}}$

$$g'(x) = 2x\sin^2(x) + x^2 \cdot \underbrace{2\sin x}_{\text{Outside}}\ \underbrace{\cos x}_{\text{inside}}$$

$$= \underbrace{2x\sin^2 x}_{\text{product rule}} + \underbrace{2x^2\sin x\cos x}_{\text{product rule}} \qquad ▨^2 \qquad \sin x$$

$$g''(x) = 2\left(1.\sin^2 x + x\,2\sin x\cos x\right) + 2\left(2x\sin x\cos x + x^2\cos x\cos x + x^2\sin x(-\sin x)\right)$$

$$= 2\sin^2 x + 4x\sin x\cos x + 4x\sin x\cos x + 2x^2\cos^2 x - 2x^2\sin^2 x$$

c) $f(x) = -4 \ln(\sin x)$

$$f'(x) = -4 \underbrace{\frac{1}{\sin x}}_{\text{outside: ln ▨}} \cdot \underbrace{\cos x}_{\text{inside}} = -4 \frac{\cos x}{\sin x}$$

$$f''(x) = -4 \frac{-\sin x(\sin x) - \cos x(\cos x)}{\sin^2 x}$$

Quotient Rule

$$= -4 \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \quad -(\sin^2 x + \cos^2 x)$$

$$= -4 \cdot \frac{-1}{\sin^2 x} = \boxed{\frac{4}{\sin^2 x}}$$

d) $y = \underbrace{2x \cos(5x)}_{\text{product}} - \underbrace{\cos^2 x}_{\text{chain}}$

$$\frac{dy}{dx} = y' = 2\cos(5x) + 2x \underbrace{(-\sin(5x))}_{\text{out}} \cdot \underbrace{5}_{\text{in}} - 2 \underbrace{(-\sin x)}_{\text{out}} \underbrace{\cos x}_{\text{in}}$$

$$= \underbrace{2\cos(5x)}_{\text{chain}} - \underbrace{10x \sin(5x)}_{\text{product}} + \underbrace{2 \sin x \cos x}_{\text{product}}$$

$$\frac{d^2y}{dx^2} = y'' = 2 \underbrace{(-\sin 5x)}_{\text{Out}} \cdot \underbrace{5}_{\text{in}} - \left[10 \sin(5x) + 10x \underbrace{(\cos 5x)}_{\text{Out}} \cdot \underbrace{5}_{\text{in}}\right]$$

$$+ 2 \cos x \cos x + 2 \sin x(-\sin x)$$

$$\boxed{= -10 \sin(5x) - 10 \sin(5x) - 50x \cos(5x) + 2\cos^2 x - 2\sin^2 x}$$

Application of Second Derivative.

Recall that for a given function $f(x)$, its first derivative $f'(x)$ represents the (instantaneous) rate of change for example velocity as the rate of change in the distance.

Following the same rule, now $f''(x)$ the second derivative would represent the rate of change in f'.

The first derivative $f'(x) \rightsquigarrow$ Rate of Change of f

The Second derivative $\underset{(f')'}{\underset{\downarrow}{f''(x)}} \rightsquigarrow$ Rate of Change of f'

For instance,

$f(t)$ is the function of distance, height, ...

$f'(t)$ is the function of velocity

$f''(t)$ is the function of acceleration

Example. The height (feet) of a particle at time t seconds is $t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when

a) The particle starts its motion

b) at $t = 1$ second.

Call the function of height for example

$$h(t) = t^3 - 4t^2 + 8t$$

$\underline{\text{height at the beginning}}$: $h(0) = 0^3 - 4\,0^2 + 8\times 0 = 0$ m

$t = 0$ NO height.

Velocity : $v(t) = h'(t) = 3t^2 - 8t + 8$

$\Rightarrow v(0) = 3\times 0^2 - 8\times 0 + 8 = \underset{m/s}{8} \rightsquigarrow$ initial velocity.

Acceleration : $a(t) = h''(t) = 6t - 8$

$a(0) = 6\times 0 - 8 = \underset{\downarrow}{-8}$ m/s^2

velocity is decreasing

Similarly : $t = 1 \rightsquigarrow h(1) = 1^3 - 4\times 1^2 + 8\times 1 = 1 - 4 + 8 = 5$ m

$v(1) = 3\times 1^2 - 8\times 1 + 8 = 3 - 8 + 8 = 3$ m/s

$a(1) = 6\times 1 - 8 = -2$ m/s^2

$\underline{\underline{\text{Example}}}$. Suppose the equation for the motion of a particle is $S(t) = \frac{1}{3}t^3 - 3t^2 + 8t + 4$

a) Find acceleration at time t.

b) Find the acceleration at the instant(s) when the velocity is 0.

(a) $$s(t) = \frac{1}{3}t^3 - 3t^2 + 8t + 4$$

$$v(t) = s'(t) = \frac{1}{3}(3t^2) - 6t + 8 = t^2 - 6t + 8$$

$$\boxed{a(t) = s''(t) = 2t - 6}$$

(b) At what instant is the velocity 0 ?

When $v(t) = t^2 - 6t + 8 = 0$

solve for t : $(t - 4)(t \quad 2) = 0$

$$\Rightarrow \quad t = 4 \quad , \quad t = 2$$

Acceleration at
$$\boxed{\begin{array}{ll} t = 4 \rightsquigarrow & a(4) = 2 \cdot 4 - 6 = 2 \ m/s^2 \\ t = 2 \rightsquigarrow & a(2) = 2 \cdot 2 - 6 = -2 \ m/s^2 \end{array}}$$

Example. Suppose a population is growing exponentially.

a) Find the rate at which the population is increasing.

b) Find the rate at which the growth rate is changing.

First : Exponential growth model :

$$y(t) = y_0 \, e^{kt}$$

y_0 : A number : initial population

k : A number : Growth rate.

a) The rate of increase in the population

$$\underbrace{\underbrace{\text{rate of increase}}_{\text{derivative}} \text{ in } \underbrace{\text{the population}}_{y(t)}}_{\text{find } y'(t)}$$

$$\underset{\text{rate of growth}}{y'(t)} = y_0 \underbrace{e^{kt}}_{\substack{\text{outside}\\ e^{\text{▨}}}} \cdot \underbrace{k}_{\text{inside}} = k \underbrace{y_0 e^{kt}}_{y(t)} = k\, y(t)$$

b) The rate of change in the growth rate

$$\underbrace{\underbrace{\text{rate of change}}_{\text{derivative}} \text{ in the } \underbrace{\text{growth rate}}_{y'(t)}}_{y''(t)}$$

$$y''(t) = \underbrace{k y_0}_{\text{a number}} \underbrace{e^{kt}}_{\text{outside}} \cdot \underbrace{k}_{\text{inside}} = k^2 \underbrace{y_0 e^{kt}}_{y(t)} = k^2 y(t)$$

Practice. Do the last example for the exponential decay model.

product rule

practice. $f(x) = \underline{e^{x}}\ \underline{\sin}\left(\frac{1}{x}\right) + 2x$

$$\boxed{\left(\frac{1}{x}\right)' = \frac{0-1}{x^{2}}}$$

$$f'(x) = \underbrace{e^{x}\sin\left(\frac{1}{x}\right)} + \underbrace{e^{x}\underbrace{\cos\left(\frac{1}{x}\right)}_{\text{outside}}\cdot\underbrace{\left(-\frac{1}{x^{2}}\right)}_{\text{inside}}} + 2$$

$$f''(x) = \underbrace{e^{x}\sin\left(\frac{1}{x}\right) + e^{x}\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^{2}}\right)}$$

product rule for 3 functions

$$+\, e^{x}\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x2}\right) + e^{x}\cdot\left(\underset{*}{-}\sin\frac{1}{x}\right)\cdot\left(\underset{*}{-}\frac{1}{x^{2}}\right)$$

$$+\, e^{x}\cos\left(\frac{1}{x}\right)\cdot\ \overset{*}{-}\,\frac{0\ \overset{*}{-}\ 2\cancel{x}}{x^{\cancel{4}3}}$$

simplify

$$= e^{x}\sin\left(\frac{1}{x}\right) - \frac{e^{x}}{x^{2}}\cos\left(\frac{1}{x}\right) - \frac{e^{x}}{x^{2}}\cos\left(\frac{1}{x}\right)$$

$$+ \frac{e^{x}}{x^{2}}\sin\left(\frac{1}{x}\right) + \frac{2e^{x}}{x^{3}}\cos\left(\frac{1}{x}\right)$$