Lecture 1 , Sep 8, 2016

- Why do we need Calculus ?

  Calculus is the study of change. It expands our understanding of how nature works.

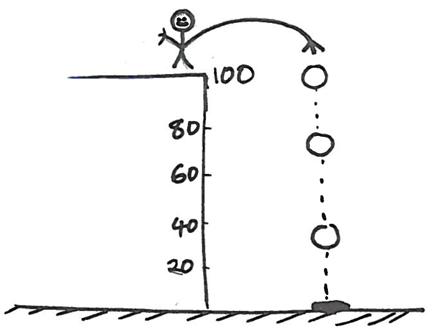
  " Algebra is about finding patterns between numbers, and Calculus finds patterns between equations. "

  Calculus equips us with some essential tools to model various natural phenomena , so that to gain information and make predictions based upon those modellings.

One of the main applications of Calculus is in study the motion of a moving object. The following example gives us an idea of why we need calculus in this particular case. We'll see this topic in detail in a few weeks.

Average Velocity vs. Instantaneous Velocity.

__Scenario__. Suppose we are on top of a 100-foot building and we drop down a ball while we are taking the time during its fall. (Assume we know the height at different points.)



time (s)	height (ft)
0	100
0.5	96
1	85
1.5	62
2	30
2.5	0

Now we want to answer the following questions.

__Q1.__ How long does it take for the ball to fall and hit the ground? 2.5 s

__Q2.__ How far does the ball fall during the 1st second?

$100 - 85 = 15$ ft ⇝ However, height is decreasing.

__Q3.__ How far does the ball fall during the last second?

$62 - 0 = 62$ ft

__Q4.__ What is the average velocity of the ball during its fall?

* To find the average velocity, we need a time interval. Then the formula is the following:

$$\text{Average velocity} = V_{av} = \frac{\text{distance travelled}}{\text{total time}}$$

$$= \frac{\text{final height} - \text{initial height}}{\text{final time} - \text{initial time}}$$

* In math, whenever we want to represent a "change" in a quantity we use the symbol (Greek letter) Δ, which is called "Delta". Note that if ◍ represents the quantity, then Δ ◍ = final ◍ − initial ◍

(final first, initial second)

Therefore, the formula for average velocity can be shortened to:

$$V_{av} = \frac{\Delta(\text{height})}{\Delta(\text{time})}$$

* Denote height by $\underline{h}$ and time by $\underline{t}$.

$$= \frac{\Delta h}{\Delta t}$$

This does NOT mean $\Delta \times t$ or $\Delta \times h$!

Now we can answer question 4:

$$V_{av} = \frac{\Delta h}{\Delta t} = \frac{[0 - 100]\ \text{ft}}{[2.5 - 0]\ \text{s}} = \frac{-100\ \text{ft}}{2.5\ \text{s}} = -40\ \text{ft/s}$$

final height

initial height

final time

initial time

Because final height is less than initial height, as the ball is falling.

Q5. What is the average velocity between $t = 1_s$ and $t = 2_s$?

$$V_{av} = \frac{\Delta h}{\Delta t} = \frac{30 - 85}{2 - 1} = \frac{-55\ ft}{1\ s} = -55\ ft/s$$

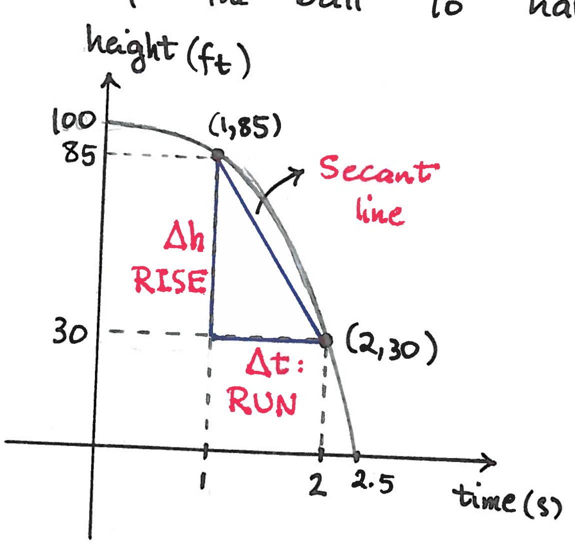
Now, what if we want to find the velocity at one instant in time and NOT in a time interval, for example:

Q6. How fast is the ball falling at $t = 1_s$?

$\downarrow$ one instant in time.

Definition. The velocity at an instant in time is called the instantaneous velocity and I denote it by V_{inst}.

To answer Q6, one way is to approximate V_{inst} with V_{av} between $t = 1_s$ and $t = 2_s$, but we would like to make our approximation more precise, so let's first graph the motion of the ball to have a better idea of the problem.



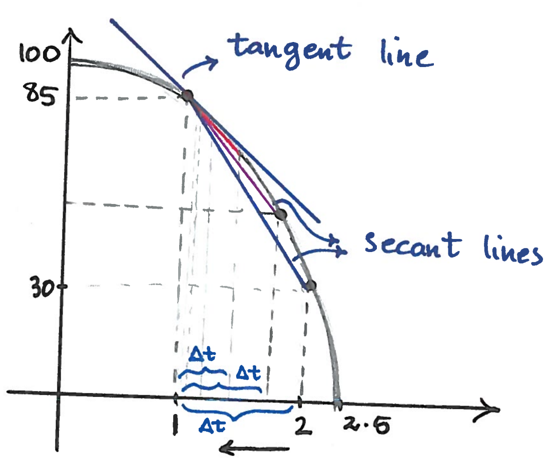
From the graph we see that

$$\boxed{V_{av} = \frac{\Delta h}{\Delta t} = \frac{\text{RISE}}{\text{RUN}}}$$

If we join the two points (1, 85) and (2, 30), then we see that average velocity is actually the slope of this line segment through these two points, which is called the secant line through the two points.

$$\Rightarrow \boxed{V_{av} = \frac{\Delta h}{\Delta t} = \frac{\text{RISE}}{\text{RUN}} = \text{Slope of the secant line}}$$

Approximating V_{inst} with V_{av} in time interval $t=1$ and $t=2$ is a crude guess, so let's make it more accurate. To do so, we start moving from $t=2_s$ to $t=1_s$ to get closer to the instant $t=1s$. Thus, we have different secant lines as we get closer and closer to $t=1$.



As we approach $t=1$, we see that the time interval, Δt, becomes smaller and smaller. This means the denominator of V_{av} formula gets smaller; and very close to $t=1$, Δt is very close to 0.

However, a 0 denominator mathematically does NOT make sense, so how can we represent $\frac{\Delta h}{\Delta t}$ then ?!

$\Longrightarrow$ Calculus helps us! :) $\Longleftarrow$

Also, as we are moving closer to $t = 1_s$, the secant line passing through two points is also changing. In fact, the two points are almost changing to one point, and the secant line is actually the tangent line at that one point on the graph, which is $(1, 85)$.

$\Rightarrow$ Instantaneous velocity $= V_{inst}$

$=$ Slope of the tangent line at the given point

? What we need to review ?

2 dimensional plane, points, slope of lines, functions etc. will be reviewed in the next few weeks. As you see we need these topics to learn the main concepts.

⇝ We'll come back to this example and formulas soon!