MATH 110 Midterm 1, October 25th, 2016
Duration: 90 minutes
This test has ***7 questions*** *on* ***10 pages****, for a total of 75 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

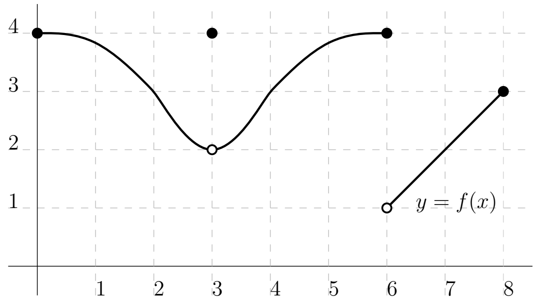
Signature: ____

Question:	1	2	3	4	5	6	7	Total
Points:	15	10	13	8	14	7	8	75
Score:								

Full-Solution Problems. In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are required**.

1. This question has 4 different problems.

$\boxed{\text{5 marks}}$ (a) Consider the function $f(x)$ whose graph is shown below:



Find the following:

(i) domain of f: $[0, 8]$

(ii) range of f: $(1, 4]$

(iii) $\lim_{x\to 3} f(x) =$ 2

Slope of the line $\leftarrow$ (iv) $f'(7) =$ $\frac{\text{Rise}}{\text{Run}} = \frac{3-1}{8-6} = 1$

(v) $f(f(4)) =$ 4

$\boxed{\text{3 marks}}$ (b) Find the domain of $f(x) = \dfrac{1}{\sqrt{x} - 1}$.

Denominator $\neq 0$ $\rightarrow$ $\sqrt{x} \neq 1 \Rightarrow x = 1$

Square Root : $x \geqslant 0$

$\Rightarrow$ Domain : $x \geq 0$ and $x \neq 1$

or : $[0,1) \cup (1, +\infty)$

3 marks (c) Consider the functions $h(x) = \dfrac{x^2+1}{x}$ and

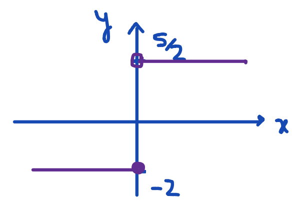
$$g(x) = \begin{cases} -1 & x \leq 0 \\ 2 & x > 0. \end{cases}$$

Find a formula for $h(g(x))$ and determine whether the composite function $h(g(x))$ is invertible. Justify your answer.

$$h(g(x)) = \begin{cases} \dfrac{(-1)^2+1}{-1} & x \leq 0 \\ \dfrac{2^2+1}{2} & x > 0 \end{cases} = \begin{cases} -2 & x \leq 0 \\ \dfrac{5}{2} & x > 0 \end{cases}$$

$h(g(x))$ NOT 1-1 $\Rightarrow$ NOT invertible

you can also check the graph to see that it fails the horizontal line test.

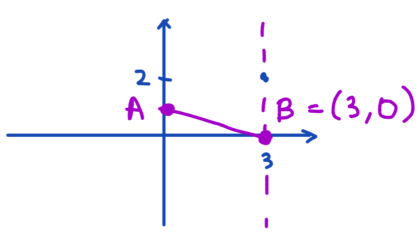


4 marks (d) Let A be the y-intercept of the line that goes through the points $(-1, 2/3)$ and $(3, 2)$. Let B be the x-intercept of the line perpendicular to the x-axis that goes through the point $(3, 2)$. Find the distance between the points A and B.

$$m = \frac{2-\frac{2}{3}}{3-(-1)} = \frac{4/3}{4} = \frac{4}{12} = \frac{1}{3}$$

Choose one point: $y - 2 = \frac{1}{3}(x-3) \Rightarrow y = \frac{1}{3}x + 1$

y-int : $y = \frac{1}{3} \cdot 0 + 1 = 1 \Rightarrow A = (0,1)$



$\Rightarrow B = (3,0)$

$$AB = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$
$$= \sqrt{(3-0)^2 + (0-1)^2}$$
$$= \sqrt{9+1} = \sqrt{10}$$

10 marks

2. Evaluate the following limits, if they exist, or specify whether they are ∞, $-\infty$, or do not exist. Show all your work.

(a) $\lim_{x\to 5} \frac{x^2-1}{x-1}$. $= \frac{5^2-1}{5-1} = \frac{24}{4} = 6$

(b) $\lim_{t\to 0} \frac{t+1}{t^2}$. $= \frac{0+1}{0^2} = \frac{1}{0^+} = +\infty$

always positive

always a small positive number

(c) $\lim_{a\to 3^-} \frac{a^2+2a+1}{a-3}$. $= \frac{3^2+2.3+1}{3^- - 3} = \frac{16}{0^-} = -\infty$

a small negative number

(d) $\lim_{x\to 2} \frac{x^2-6x+8}{x^2-3x+2}$. $= \frac{2^2-6\cdot 2+8}{2^2-3.2+2} = \frac{0}{0} \longrightarrow$ Factorize

$= \lim_{x\to 2} \frac{(x-4)\cancel{(x-2)}}{\cancel{(x-2)}(x-1)} = \frac{2-4}{2-1} = -2$

(e) $\lim_{t\to -2} \frac{1+t^2}{2+t}$. $= \frac{1+(-2)^2}{2+(-2)} = \frac{5}{0} = \begin{cases} \lim_{t\to -2^-} \cdots = \frac{5}{0^-} = -\infty \\ \lim_{t\to -2^+} \cdots = \frac{5}{0^+} = +\infty \end{cases}$

positive or negative ?

limit DNE

3. This question has 3 different problems.

4 marks (a) Find y' if $y = 2x^5 + \sqrt[3]{x^2} - \dfrac{3}{\sqrt{x^3}}$. $= 2x^5 + x^{\frac{2}{3}} - 3x^{-\frac{3}{2}}$

Power Rule: $y' = 10x^4 + \frac{2}{3}x^{\frac{2}{3}-1} - 3(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$= 10x^4 + \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-\frac{5}{2}}$

Simplification is optional. $\longleftarrow$ $= 10x^4 + \dfrac{2}{3\sqrt[3]{x}} + \dfrac{9}{2x^2\sqrt{x}}$

5 marks (b) Find the equation of a line tangent to the graph of $y = x^3 - 3x$ at the point of x-coordinate $x = 2$.

point $x=2 \Rightarrow y = 2^3 - 3.2 = 2 \Rightarrow (2,2)$

slope $m = f'(2) = y'(2)$

$y'(x) = 3x^2 - 3 \Rightarrow y'(2) = 3(2)^2 - 3 = 9$

$y - y_1 = m(x - x_1) \Rightarrow y - 2 = 9(x-2)$

$\Rightarrow y = 9x - 16$

4 marks (c) Find a value for the constant a such that the graph of $y = ax^3$ is tangent to the line $y = 3x + 1$ at some point.

The tangency point is NOT given, but we know it's on f so we have the coordinates for that: $P = (x, ax^3)$

tangent line $y = \underbrace{3}_{m_{tan}}x + 1$, but $m_{tan} = f'(x)$

$3 = 3ax^2 \Rightarrow ax^2 = 1 \Rightarrow a = \dfrac{1}{x^2}$

tangency point is both on line and on the graph $\rightsquigarrow$ it must satisfy both equations.

$y = 3x+1$
$y = ax^3$ $\rightsquigarrow$ $3x + 1 = ax^3 = \underbrace{ax^2}_{1}\cdot x$

$\Rightarrow 3x + 1 = x \Rightarrow 2x = -1 \Rightarrow x = \dfrac{-1}{2}$

$\Rightarrow a = \dfrac{1}{x^2} = \dfrac{1}{(-\frac{1}{2})^2} = \dfrac{1}{\frac{1}{4}} \boxed{= 4}$

8 marks 4. Consider the function

$$f(x) = \begin{cases} x^2 - x - 6 & x < 0 \\ -6 & x > 0. \end{cases}$$

Answer the questions below. No marks will be given to answers without justification.

(a) Graph $f(x)$. Make sure you mark the intercepts of the function with the coordinate axes, if they exist, by indicating their value on the axes.

$y = x^2 - x - 6$

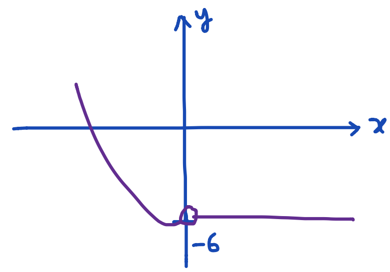
y-int : $(0, -6)$

x-int : $x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3,\ x = -2$

vertex : $x = \frac{-b}{2a} = \frac{1}{2}$,



(b) Is there a number a for which the limit $\lim_{x\to a} f(x)$ does not exist? Justify your answer.

Check left & right limits : (The only point to check is 0).

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 - x - 6 = -6$

$\lim_{x\to 0^+} f(x) = -6$

Equal $\longrightarrow$ limit exists everywhere $\Rightarrow$ NO such "a" exists.

For limit we do NOT care about $f(0)$

(c) Is there a number a such that $f(x)$ is not continuous at $x = a$? Justify your answer.

Check $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

However, $f(0)$ is NOT defined in the equation. (Hole in the graph) $\Rightarrow$ NOT cont's at $x=0$.

(d) Is there a number b such that the following piecewise function

$$g(x) = \begin{cases} f(x) & x < 0 \\ b & x = 0 \\ -6 + x & x > 0 \end{cases}$$

From above $f(x) = x^2 - x - 6$

is continuous everywhere? Justify your answer.

To have continuity everywhere

$\underbrace{\lim_{x\to 0^-} g(x)}$ $=$ $\underbrace{\lim_{x\to 0^+} g(x)}$ $= g(0)$

$\lim_{x\to 0^+} -6 + x = -6$

$\lim_{x\to 0^-} x^2 - x - 6 = -6$

$\boxed{g(0) = b = -6}$

14 marks 5. An outdoor hot tub holds $4000L$ of water. If a small valve at the bottom of the tub is opened, then the volume of water in the tub is modelled by the function

$$V(t) = 4000(1-t)^2,$$

where V is the volume of water in the hot tub, in *litres*, and t is the time, in *hour*, since the valve is open.

(a) How long does it take for the water to completely drain?

$V(t) = 0$

$\Rightarrow 4000(1-t)^2 = 0 \Rightarrow (1-t)^2 = 0$

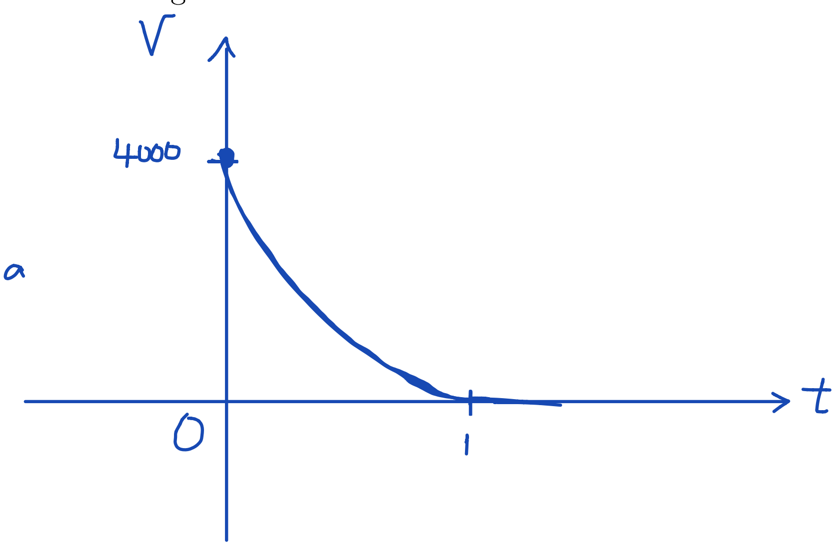
$\Rightarrow t = 1$ hour

(b) Sketch the graph of the volume function. (Note that by part (a) there would be no water in the tub after some point in time.) Make sure you indicate the value of any intercept on your sketch. Your diagram does not need to be in scale.

$t=0 \rightsquigarrow V(t) = 4000$

$t=1 \rightsquigarrow V(t) = 0$

quadratic function $\rightsquigarrow$ Parabola



(c) Find the average rate of change in the volume of the water during the first half hour. Include units in your answer.

$[0, \frac{1}{2}]$

$$R_{av} = \frac{V(\frac{1}{2}) - V(0)}{\frac{1}{2} - 0}$$

$$= \frac{4000(1-\frac{1}{2})^2 - 4000(1)^2}{\frac{1}{2}}$$

$$= \frac{4000(\frac{1}{4} - 1)}{\frac{1}{2}} = 4000 \frac{-\frac{3}{4}}{\frac{1}{2}} = 4000 \cdot -\frac{3}{2} = -6000 \frac{\text{lit}}{\text{hour}}$$

(d) Using the definition of instantaneous rate of change as a limit, compute the instantaneous rate of change of V at $t = 1/2$ hour.

$\longrightarrow V'(\tfrac{1}{2})$

$$V'(\tfrac{1}{2}) = \lim_{h\to 0} \frac{V(\frac{1}{2}+h) - V(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{4000\left(1-(\frac{1}{2}+h)\right)^2 - 4000\left(\overset{\frac{1}{4}}{\cancel{1-\frac{1}{2}}}\right)^2}{h}$$

$$= \lim_{h\to 0} \frac{4000(\frac{1}{2}-h)^2 - 1000}{h}$$

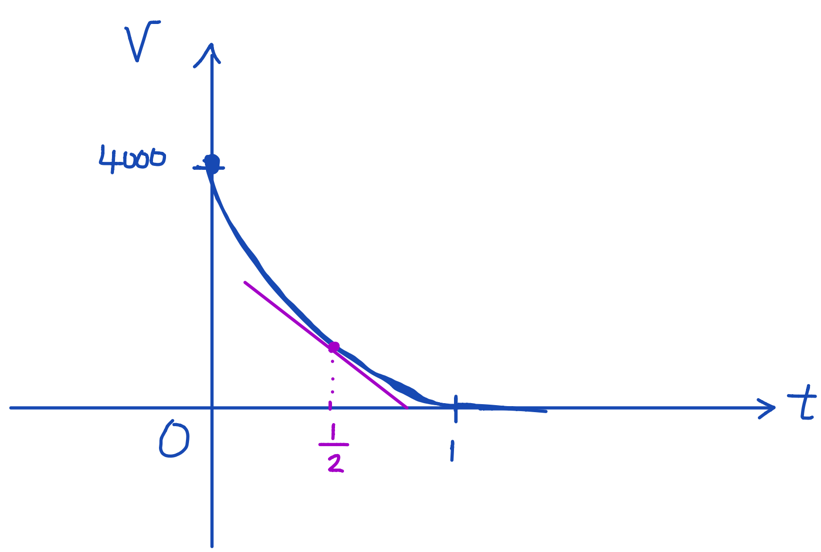
$$= \lim_{h\to 0} \frac{4000(\frac{1}{4}-h+h^2) - 1000}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1000} - 4000h + 4000h^2 - \cancel{1000}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-4000+4000h)}{\cancel{h}}$$

$$= -4000 \ \frac{\text{lit}}{\text{hour}}$$

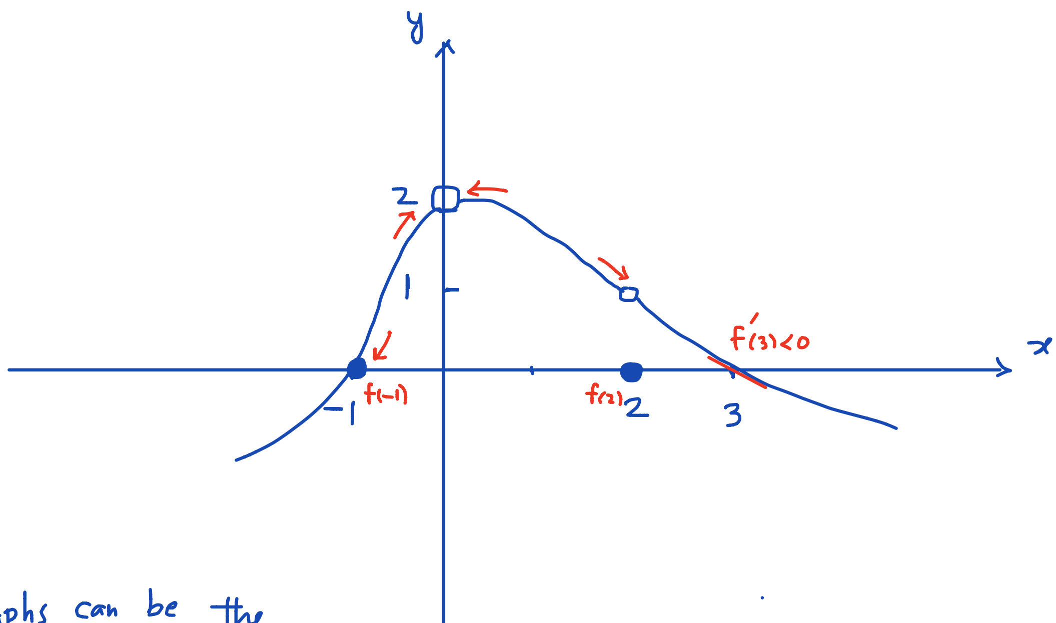
(e) Determine the slope of the tangent line to the graph of $V(t)$ at $t = 1/2$ hour. Then sketch the tangent line on your graph created in part (b) (or reproduce the graph below).



7 marks 6. Sketch the graph of a function f satisfying all of the following properties:

1. The domain of the function is all real numbers except 0,
2. $\lim_{x \to -1^+} f(x) = 0$,
3. $\lim_{x \to 0} f(x) = 2$,
4. $\lim_{x \to 2^-} f(x) = 1$,
5. $f(-1) = f(2)$,
6. $f'(3) < 0$.

Make sure you clearly identify the points on the graph that are related to the above conditions by indicating the value of their coordinates on the axes. Your graph does not need to be in scale. *Note:* You are NOT required to provide a formula for $f(x)$.



Different graphs can be the answer to this question.

8 marks 7. (a) State the Intermediate Value Theorem for a function f defined on an interval $[a, b]$. Make sure you clearly state the assumptions and the conclusion of the theorem.

If f is continuous on $[a,b]$ and $f(a) \neq f(a)$, and if N is a value between $f(a)$ and $f(b)$, then there is a number c in (a,b) such that $f(c) = N$.

(b) Use the Intermediate Value Theorem to show that there is a point x in the interval $(0, 1)$ at which the graphs of the functions f and g intersect, where

$$f(x) = x^3 + 2x, \quad g(x) = x^2 + 1.$$

Make sure you justify your claims.

Two functions intersect when $f(x) = g(x) \Rightarrow f(x) - g(x) = 0$

Denote $h(x) = f(x) - g(x) = x^3 + 2x - x^2 - 1$

We want to show that $h(x) = 0$ has a solution.

$h(x)$ is continuous everywhere (a polynomial)

and $h(0) = -1 < 0$

$h(1) = 1 > 0$

Therefore, by IVT, there must be a number c in $(0,1)$ such that

$h(c) = 0$

So $f(c) = g(c) \rightsquigarrow f$ and g intersect at a point c in $(0,1)$.