Exponential Decay :

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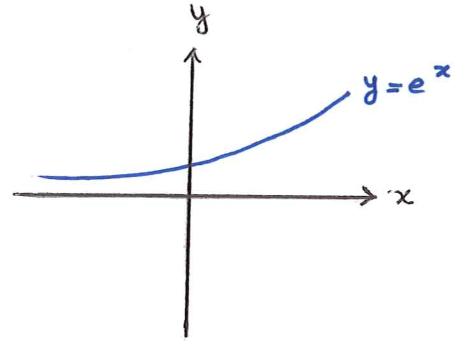
Lec 23

Recall Exponential functions:

$y = e^x$ always positive

$y' = e^x$ always positive
$\rightsquigarrow$ always increasing



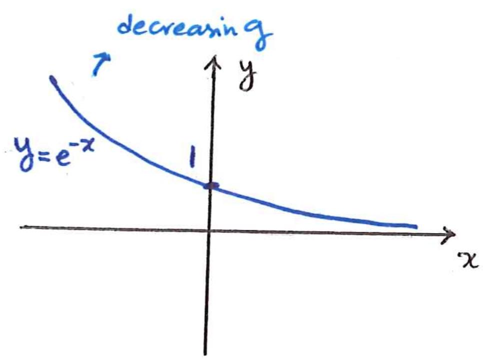
Domain : $(-\infty, +\infty)$, x-int : None
Range : $(0, +\infty)$, y-int : 1

Now we want to see how the function $y = e^{-x}$ looks, and what the differences are.

$y = e^{-x}$ still always positive because $e^{\square}$ is always positive.

But

$y' = -e^{-x}$ Always negative
$\rightsquigarrow$ always decreasing



Domain : $(-\infty, +\infty)$, x-int : None
Range : $(0, +\infty)$, y-int : 1

$\overset{x=0}{\rightsquigarrow} y(0) = e^{-0} = e^0 = 1$

Therefore a function which looks like $y = e^{-t}$ can be a good choice to model behaviours that decay exponentially.

In fact, the decay model is very similar to growth model, except that we need to change the function so that it becomes decreasing.

Exponential Decay Model — $\boxed{y(t) = y_0 e^{-kt}}$

- $y(t)$: The quantity that's decaying. like population, depreciation of an object, ...
- y_0: initial value of the quantity
- k: Rate Constant. A positive number that determines the growth Rate.

Similar Observations:

- **Decay Rate** ⇝ Rate of decrease in the quantity ⇝ Derivative: $y'(t)$

$$y'(t) = y_0 e^{-kt} \cdot -k$$

$$\Rightarrow \boxed{y'(t) = -k\, y(t)} \longrightarrow \text{Rate of Decay}$$

Similar to growth, the rate of decay is proportional to the value of quantity at any time t.

- **Half-life**: It does NOT make sense to expect a doubling time for a decaying quantity, so we define the new terminology to be the time required for the quantity to be half.

We start with $y_0 \xrightarrow{t=?} \frac{1}{2} y_0$

We use the model to compute t.

$$y(t) = y_0 e^{-kt}$$

$$\frac{1}{2} y_0 = y_0 e^{-kt}$$

$$\xrightarrow[y_0]{\text{divide by}} \quad \frac{1}{2} = e^{-kt}$$

$$\xrightarrow[\ln]{\text{take}} \quad \ln\left(\frac{1}{2}\right) = \cancel{\ln}\, \cancel{e}^{-kt} = -kt$$

$$\xrightarrow[\text{for } t]{\text{solve}} \quad t = \frac{\ln\left(\frac{1}{2}\right)}{-k} = \frac{\ln 2^{-1}}{-k}$$

$$= \frac{-\ln 2}{-k}$$

Doubling time ← $\boxed{T_{\frac{1}{2}} = \frac{\ln 2}{k}}$ → Half-life

* Doubling time and Half-life have the same formula.

Exponential decay model has its application in determining the age of ancient object by observing and measuring the amount of decayed carbon in the object.

One other application is in determining the amount of drug remained in the blood as the drug is absorbed.

The following two examples are about these applications.

Application of Exponential Decay

Carbon Dating Estimating the age of ancient objects such as fossils, bones, meteorites, ...

When a living organism dies, it stops using carbon C-14. Researches compare the C-14 present in a living organism to the amount in a dead sample, and using the fact that the C-14 half life is $T_{\frac{1}{2}} = 5730$ years, they estimate the age of the object.

Example. Researchers determine that a fossilized bone has 30% of the C-14 of a live bone. Assuming a half-life for C-14 is 5730 years, estimate the age of the bone?

$y(t)$ = the amount of Carbon that is decaying over time

$$y_0 \xrightarrow{?\, t \text{ years}} 30\%\ y_0 = 0.3\, y_0$$

$$y(t) = y_0\, e^{-kt}$$

$$0.3\, y_0 = y_0\, e^{-kt} \xrightarrow[\text{by } y_0]{\text{divide}} 0.3 = e^{-kt} \quad (*)$$

We don't have k, so we use the half-life info to find k, then insert k into the equation $(*)$ and solve for t.

Half-life: $T_{\frac{1}{2}} = \frac{\ln 2}{k} \rightsquigarrow 5730 = \frac{\ln 2}{k}$

$$\Rightarrow \quad 5730\, k = \ln 2$$

$$\Rightarrow \quad k = \frac{\ln 2}{5730}$$

Now go to (*):

$$0.3 = e^{-\frac{\ln 2}{5730} t}$$

take ln

$$\ln 0.3 = \ln e^{-\frac{\ln 2}{5730} t}$$

cancel ln & e

$$\ln 0.3 = -\frac{\ln 2}{5730} t$$

$$\Rightarrow \quad t = \frac{\ln 0.3}{-\frac{\ln 2}{5730}} \approx 9950 \text{ years.}$$

Note: Since $0.3 < 1 \Rightarrow \ln 0.3$ is a negative number and with the other negative sign we get a positive answer.

Drug Absorption

The process of elimination of most drugs from the body can be modeled by an exponential decay function with a known half-life.

Example. An exponential decay function models the amount of drug in the blood with an initial dose of $y_0 = 100$ mg. Assume the half-life of the drug is 16 hours.

a) Find the exponential decay function that governs the amount of drug in the blood.

b) How much time is required for the drug to reach 1% of the initial dose?

c) If a second 100 mg dose is given 12 hours after the first dose, how much time is required for the drug level to reach 1 mg?

a) $y(t) = y_0 e^{-kt} \rightsquigarrow$ Model

$y_0 = 100$ mg , we must find k with the half-life info.

$$T_{\frac{1}{2}} = \frac{\ln 2}{k} \Rightarrow 16_{hrs} = \frac{\ln 2}{k}$$

$$\Rightarrow 16k = \ln 2 \Rightarrow k = \frac{\ln 2}{16}$$

$$\Rightarrow \boxed{y(t) = 100\, e^{-\frac{\ln 2}{16} t}}$$

b) start with $y_0 = 100$ mg $\xrightarrow{?\ t \text{ hours}}$ $1\%\, y_0 = 0.1\, y_0 = 1$ mg

$$1 = 100\, e^{-\frac{\ln 2}{16} t}$$

$$\Rightarrow \frac{1}{100} = e^{-\frac{\ln 2}{16} t}$$

$$\underset{\ln}{\overset{\text{take}}{\Rightarrow}} \ln\left(\frac{1}{100}\right) = \cancel{\ln}\, \cancel{e}^{-\frac{\ln 2}{16} t}$$

$$\Rightarrow \ln\left(\frac{1}{100}\right) = -\frac{\ln 2}{16} t$$

$$\Rightarrow \boxed{t = \frac{\ln\left(\frac{1}{100}\right)}{-\frac{\ln 2}{16}} \approx 106 \text{ hours}}$$

More than 4 days.

c) start with $100_{mg} \xrightarrow{12 \text{ hours}} \underbrace{y(12)}_{\text{What remains after 12 hrs from the initial amount}} + \underbrace{100_{mg}}_{\text{Another } 100\,mg} \xrightarrow{t \text{ hrs } ?} 1\ mg$

First use the model to find $y(12)$.

$$y(t) = 100\, e^{-\frac{\ln 2}{16} t}$$

$$\Rightarrow y(12) = 100\, e^{-\frac{\ln 2}{16} \cdot 12} \approx 59.5_{mg}$$

Now add 100_{mg} to $y(12)$:

$$\text{New amount} = 100 + 100\, e^{-\ln 2 \cdot \frac{3}{4}} \approx 159.5_{mg}$$

Now we start with $\underbrace{100 + 100\, e^{-\ln 2 \cdot \frac{3}{4}}}_{mg} \xrightarrow{?\ t_{hrs}} 1\ mg$

Again use the model:

$$y(t) = 159.5 e^{-\frac{\ln 2}{16} t} = \left(100 + 100\, e^{-\ln 2 \cdot \frac{3}{4}}\right) e^{-\frac{\ln 2}{16} t}$$

$$\Rightarrow 1_{mg} = 159.5_{mg}\, e^{-\frac{\ln 2}{16} t}$$

$$\Rightarrow \frac{1}{159.5} = e^{-\frac{\ln 2}{16} t}$$

A negative number

Recall: $\ln \frac{1}{159.5} = \ln (159.5)^{-1} = -\ln 159.5$

take ln

$$\Rightarrow \ln \frac{1}{159.5} = -\frac{\ln 2}{16} t \Rightarrow \boxed{t = \frac{\ln \frac{1}{159.5}}{-\frac{\ln 2}{16}} = 117.1_{hrs}}$$

Practice. The number of frogs in a pond grows at a rate proportional to their population, and it triples every 5 hours. = Exponential Growth

(a) How long will it take to have a 100% increase in the population?

(b) If it takes 15 hours for the population to become 3^6 frogs, what is the initial number of frogs?

(a) 100% increase means population gets doubled so we must find

$\rightsquigarrow$ Doubling time.

doubling time $\leftarrow T_2 = \dfrac{\ln 2}{k}$, But we don't have k. How should we find k? Use other info: triples every 5 hours.

start with $y_0 \xrightarrow{t=5\text{ hrs}} 3y_0$

By model: $y(t) = y_0 e^{kt}$

$t=5 \downarrow$

$$3y_0 = y_0 e^{5k}$$

divide by y_0

$$3 = e^{5k}$$

$\xrightarrow[\ln]{\text{take}}$ $\ln 3 = \cancel{\ln}\, \cancel{e}^{5k}$

$\Rightarrow \ln 3 = 5k \Rightarrow \boxed{k = \dfrac{\ln 3}{5}}$

Now doubling time : $T_2 = \dfrac{\ln 2}{k} = \dfrac{\ln 2}{\frac{\ln 3}{5}} \boxed{= 5\dfrac{\ln 2}{\ln 3}}$

(b) Unknown : y_0

info : $y_0 \xrightarrow{t = 15\,hr} \underbrace{3^6}_{y(15)}$ frogs

Model : $y(t) = y_0 e^{\frac{\ln 3}{5} t}$

$\Rightarrow 3^6 = y_0 e^{\frac{\ln 3}{\cancel{5}_1} \cdot \cancel{15}^3}$

$\Rightarrow 3^6 = y_0 e^{3 \ln 3}$

to cancel e and ln they must sit exactly next to each other.

take 3 up

$\Rightarrow 3^6 = y_0 \cancel{e}^{\cancel{\ln} 3^3}$

cancel e & ln

$\Rightarrow 3^6 = y_0 3^3$

divide by 3^3

$\Rightarrow y_0 = \dfrac{3^6}{3^3} = 3^3 = 27$ frogs were initially in the pond.

<u>Practice</u>. A scientist is studying a sample of a rare chemical element. With great effort he has produced a sample of 100% purity. The next day, 17 hours later he comes back to his lab and discovers that his sample is 37% pure.

What is the half-life of the element?

half-life $T_{\frac{1}{2}} = \dfrac{\ln 2}{k}$, what is k ?

$y(t) = y_0 e^{-kt}$ start with $y_0 \xrightarrow{t=17\text{hrs}} 37\%\, y_0$

$= 0.37\, y_0$

$\overset{t=17}{\Rightarrow} \quad 0.37\, y_0 = y_0 e^{-k\,17}$

$\Rightarrow \quad 0.37 = e^{-17k} \underset{\ln}{\overset{\text{take}}{\Rightarrow}} \ln 0.37 = -17k$

$\Rightarrow \quad k = \dfrac{\ln 0.37}{-17}$

Is this number positive?

$$\Rightarrow \quad T_{\frac{1}{2}} = \frac{\ln 2}{\frac{\ln 0.37}{-17}} = \boxed{-\frac{17 \ln 2}{\ln 0.37}}$$