Derivative of a Function

Oct 11

Lec 10

In lecture 5, we found the instantaneous velocity of a ball by estimating it with average velocity, in small time intervals. By using the table of values we tracked V_{av} in small time intervals around $t=1$ and we concluded

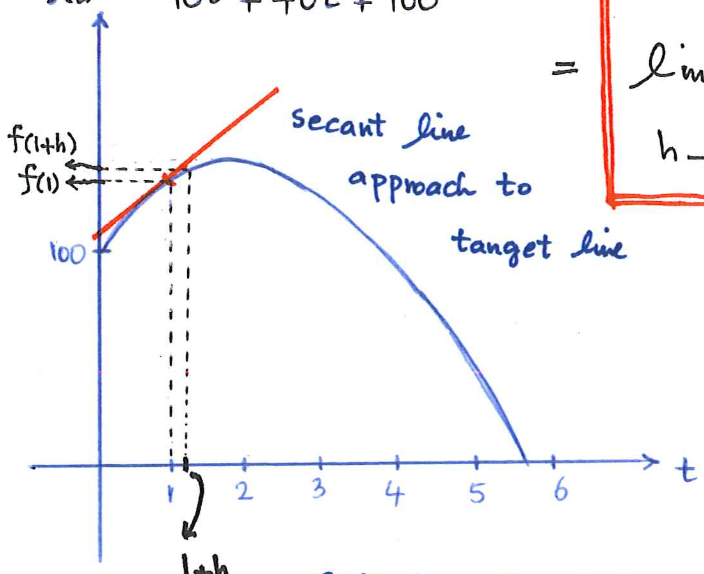
$V_{inst} = 20$ at $t=1$.

Graphically, we know V_{av} is slope of the secant line, and as we get closer to $t=1$, secant line approaches to tangent line and thus

$$\underset{\text{at } t=1}{V_{inst}} = \text{slope of tangent line at } t=1$$

$$= \boxed{m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}}$$

$f(t) = -10t^2 + 40t + 100$



$m_{sec} \longrightarrow m_{tan}$

get close to $t=1$ ⇝ $h \to 0$

By using the limit formula, this time we find V_{inst} NOT with estimation; But with limit computation.

Example 1. [Find the inst. velocity (V_{inst}) of the following function (motion of the ball) at $t=1$

equivalent

Find the slope of tangent line to the following function at $t=1$.

$$f(t) = -10t^2 + 40t + 100$$

$$\boxed{V_{inst} = m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}}$$

We use the equation of $f(t)$:

$$m_{tan} = V_{inst} = \lim_{h \to 0} \frac{\overbrace{\left[-10(1+h)^2 + 40(1+h) + 100\right]}^{f(1+h)} - \overbrace{\left[-10(1)^2 + 40 \times 1 + 100\right]}^{f(1)}}{h}$$

$$= \lim_{h \to 0} \frac{-10(1+2h+h^2) + 40 + 40h + 100 - 130}{h}$$

$$= \lim_{h \to 0} \frac{-10h^2 - 20h + 40h}{h}$$

[1^{st} step: substitution $0/0$]

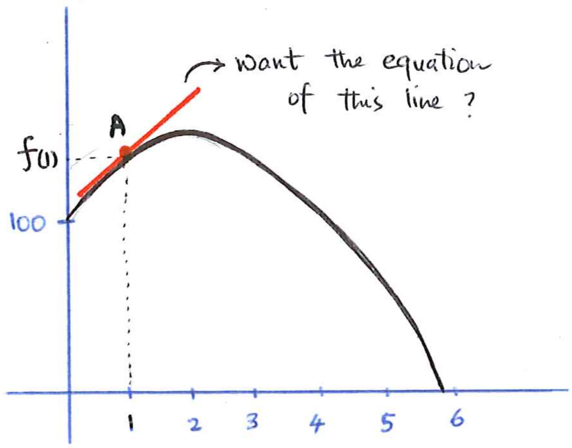
$$= \lim_{h \to 0} \frac{-10h^2 + 20h}{h}$$

Factorize

$$= \lim_{h \to 0} \frac{\cancel{h}(-10h + 20)}{\cancel{h}} = -10 \times 0 + 20 = \boxed{20}$$

what we had from table.

Example 2. In the last example, find the equation of the tangent line at $t = 1$?



To find the equation of any line we need to find its slope and a point on the line.

To find the slope m_{tan}, we have: $m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

to find a point, we know $(1, f(1)) = (1, 130)$ is the touch (tangency) point A. This point lies both on the line and on the graph of the function.

$m_{tan} = 20$ (From Ex 1.)

Point $= (1, 130)$

line equation : $y - y_1 = m(x - x_1)$

$y - 130 = 20(x - 1)$

$\Rightarrow y = 20x - 20 + 130$

$\Rightarrow \boxed{y = 20x + 110}$ tangent line equation

Derivative of $f(x)$ at $x = a$

$\underbrace{\text{f prime of a}}_{\text{Read:}}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Notation

By the last example, we see that each of the following are equivalent:

(inst.) Rate of Change at $x = a$

Slope of tangent line at $x = a$

m_{tan}

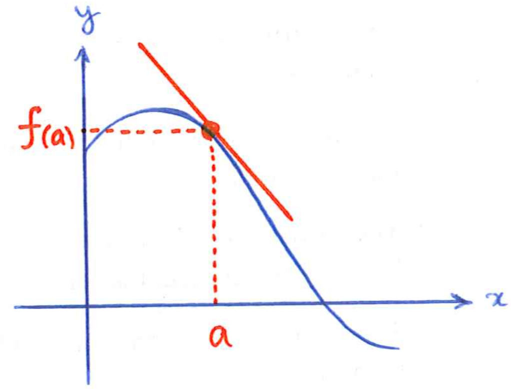
Derivative of f at $x = a$

$f'(a)$

$$m_{tan} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

What we found in Lecture 5 as $V_{inst} = 20$ and also in example on is just $\boxed{f'(1) = m_{tan} = V_{inst} = 20}$

Equation of the tangent line at $x = a$



$$\begin{cases} \text{Slope} = m_{tan} = f'(a) \\ \text{Point} = (\underbrace{a}_{x_1}, \underbrace{f(a)}_{y_1}) \end{cases}$$

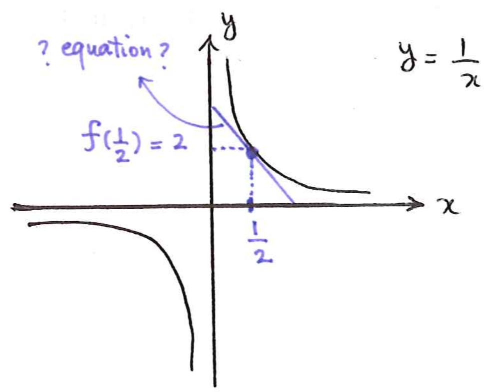
Equation:

$$y - f(a) = f'(a)(x - a)$$

$y - y_1 = m(x - x_1)$

Example 3. Find the equation of the line tangent to the graph of $y = \frac{1}{x}$ at $x = \frac{1}{2}$?

Graphically



Point ⤳ tangency point

$x = \frac{1}{2} \Rightarrow y = \frac{1}{\frac{1}{2}} = 2$

$(\frac{1}{2}, 2)$

$$\text{Slope} = m_{tan} = f'\left(\tfrac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{\frac{1}{2}+h} - 2}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1 - 2\left(\frac{1}{2}+h\right)}{\frac{1}{2}+h}}{\dfrac{h}{1}}$$

x x

$$= \lim_{h\to 0} \frac{\overset{0}{\cancel{1 - 1}} - 2h}{h\left(\frac{1}{2}+h\right)}$$

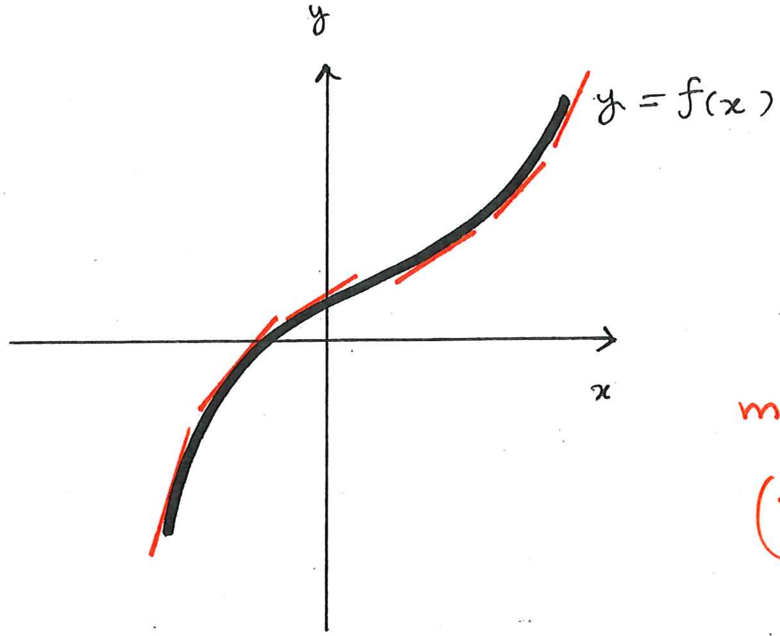
$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}\left(\frac{1}{2}+h\right)} = \frac{-2}{\frac{1}{2}+0}$$

$$= \frac{-2}{\frac{1}{2}} = \boxed{-4} \; f'\left(\tfrac{1}{2}\right)$$

$m_{tan} = -4$

Point $= \left(\frac{1}{2}, 2\right)$ $\quad y - 2 = -4\left(x - \frac{1}{2}\right)$

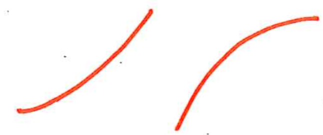
$$\boxed{y = -4x + 4}$$



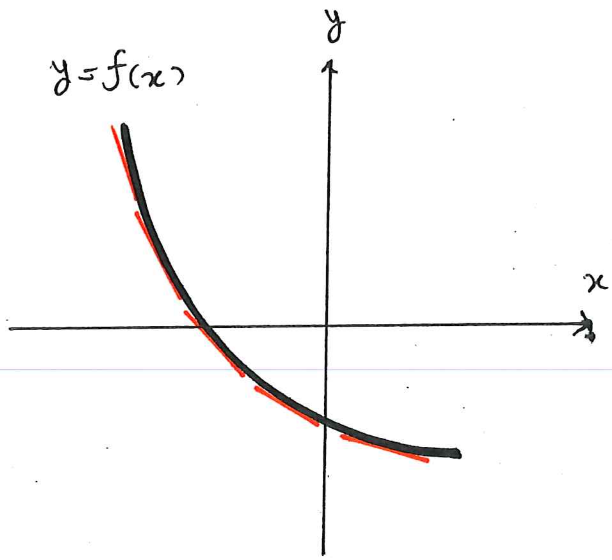
Q. Find a relation between function graph and the slope of the tangent line at different points.

$m_{tan} > 0 \rightsquigarrow f'$ positive

(positive slope) $\Rightarrow$ Graph

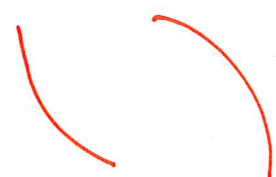


INCREASING



$m_{tan} < 0 \rightsquigarrow f'$ negative

(negative slope) $\Rightarrow$



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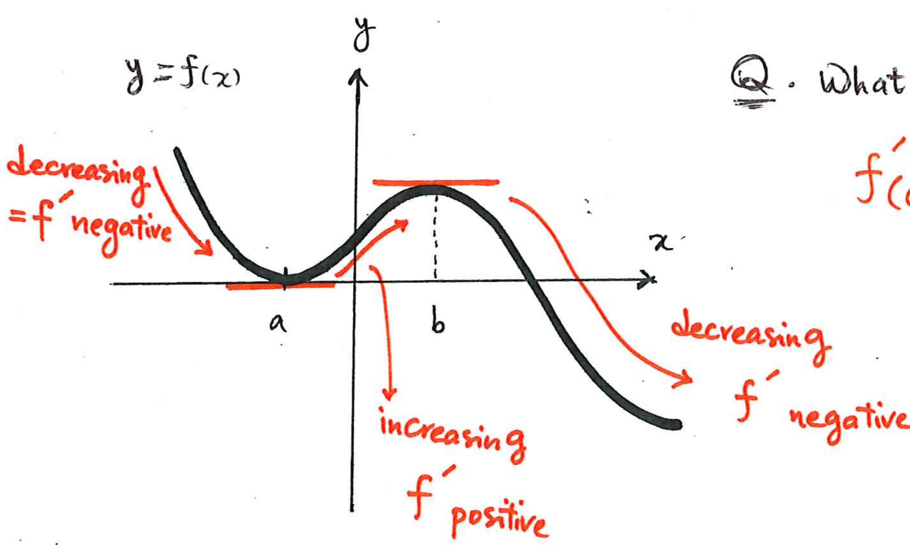

Q. What is $f'(a)$ and $f'(b)$?

$f'(a) = 0 = f'(b)$

Horizontal tangent line slope $= 0$