Differentiability

Oct 20

Lecture 13

Recall that for the function $f(x)$, the derivative of f at $x=a$ is defined by the following limit:

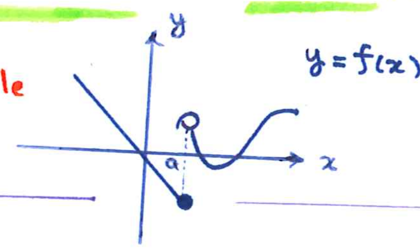
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \circledast$$

Definition: We say that f is differentiable at $x=a$ when the limit $\circledast$ above exists.

⇝ When does differentiability fail?

1) If f is discontinuous at $x=a$ then f is NOT differentiable at "a".

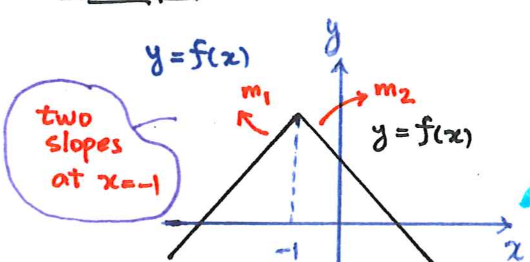
NOT Cont's $\Longrightarrow$ NOT diff'able at "a"



Question? Are continuous functions differentiable?

NO, not necessarily.

Example.



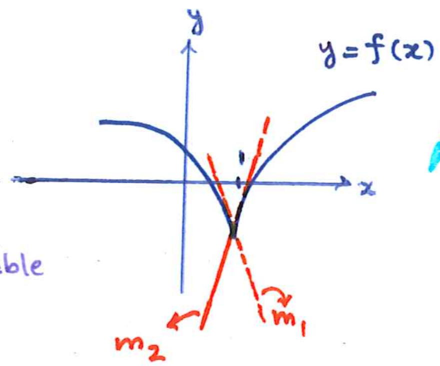
f is NOT diff'able at $x=-1$, But it is cont's.

f is cont's everywhere. But at $x=-1$ from left and from right, there are two different tangent lines, two slopes $\Rightarrow f'(-1)$ Does NOT exist.

2) If f has a corner in its graph

$\Rightarrow f$ is NOT differentiable at the corner point.

Example.



f NOT diff'able at $x=1$

But cont's.

two tangent lines at $x=1$

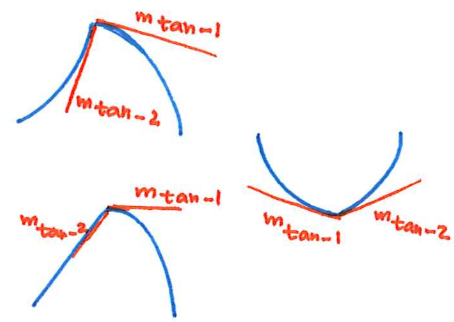
f cont's at $x=1$

But there are again two tangent line, so NOT a unique m_{tan}

$\Rightarrow f'(1)$ Does NOT Exist

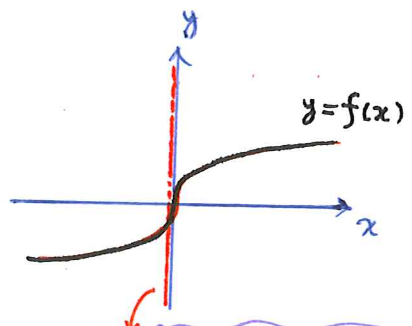
3) If f has a cusp in its graph

$\Rightarrow f$ NOT diff'able



"two tangent lines!"

Example.



f NOT diff'able at $x=0$

vertical tangent line

m_{tan} = undefined

f is cont's at $x=0$

But at $x=0$, the tangent line is vertical, so m_{tan} is NOT defined

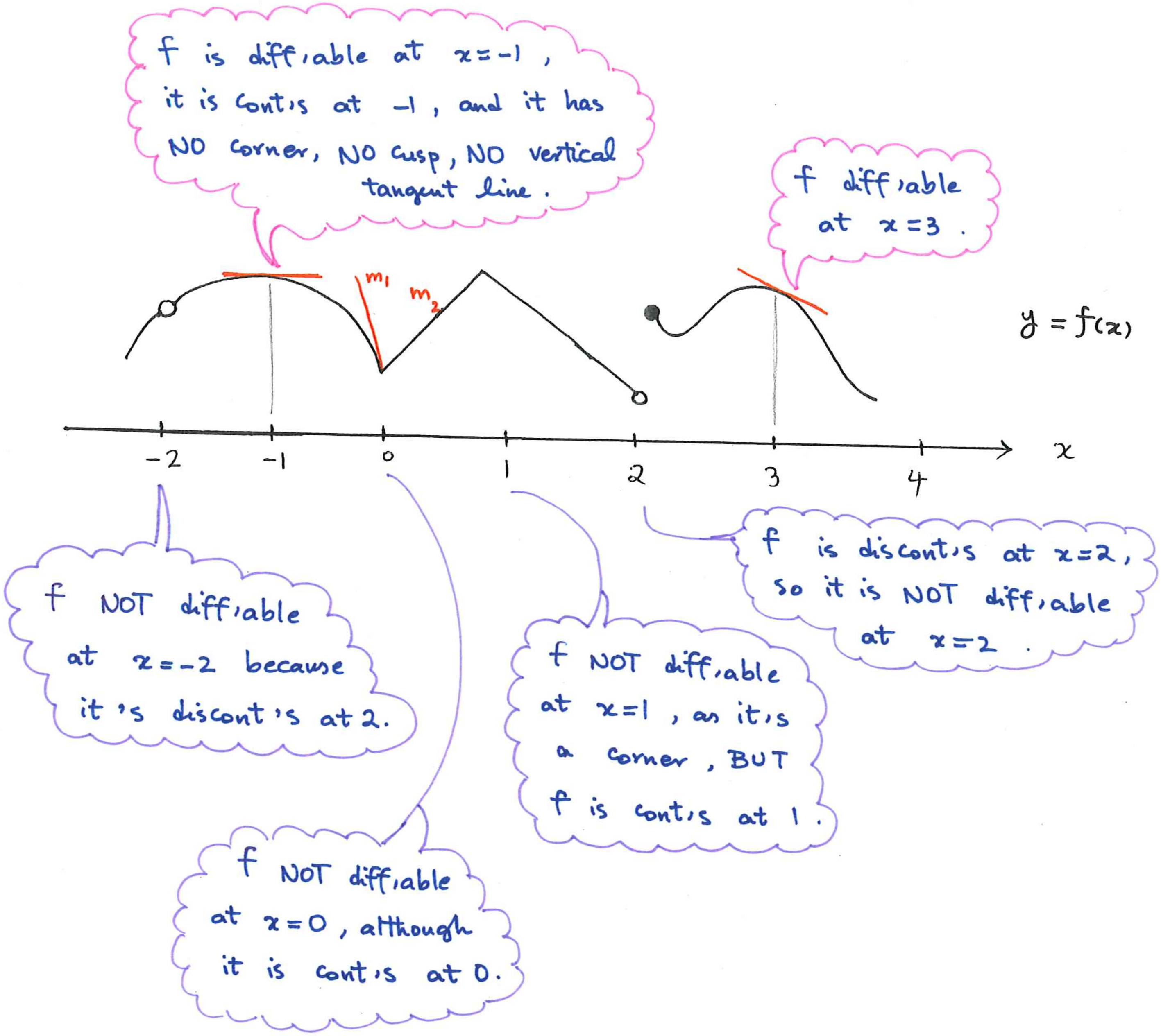
$\Rightarrow f'(0)$ Does NOT Exist.

4) If f has a vertical tangent line at $x=a$

$\Rightarrow f$ NOT diff'able at $x=a$

Graphically :

Example



f is SMOOTH!

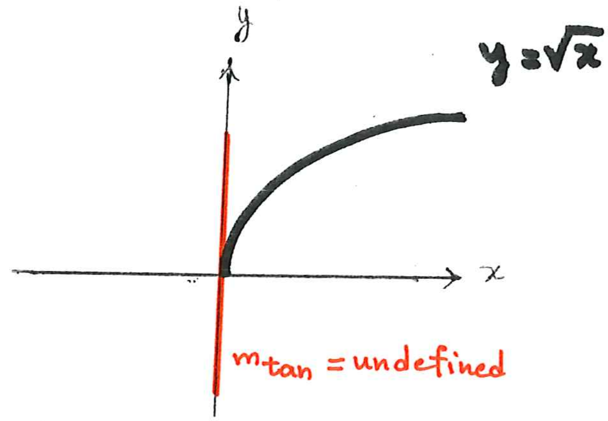
f discont's at $x=a$ $\Longrightarrow$ f NOT diff'able at $x=a$

f Cont's at $x=a$
+ NO corner
+ NO cusp
+ NO vertical tangent line

$\Longrightarrow$ f diff'able at $x=a$

Example.

$$y = \sqrt{x}$$



From Graph:

At $x=0$, there is a vertical tangent line to the graph, so m_{tan} = undefined and thus the function is NOT diff'able at $x=0$

From Equation:

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

We find the derivative of f by using power rule, and see how the derivative, $f'(x)$, behaves at $x=0$.

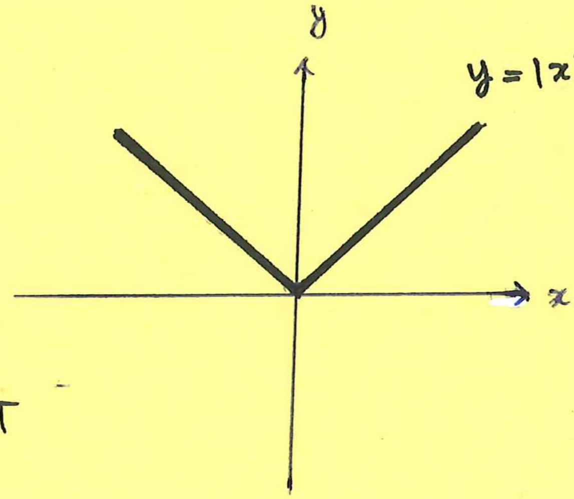
$$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2x^{1/2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$f'(0) = \frac{1}{2\sqrt{0}} = \frac{1}{0} = \text{undefined} \Rightarrow \text{NOT diff'able at } x=0.$$

Practice. $y = |x|$



Use the limit definition to show that at $x=0$, $y=|x|$ is NOT differentiable $\rightsquigarrow$ $f'(0)$ Does NOT Exist.

From the Graph, f has a corner at $x=0$ and thus it's NOT diff'able at $x=0$.

From the definition of $f'(0)$: $\quad f(x) = |x|$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

Recall: $|h| = \begin{cases} h & h \geq 0 \\ -h & h < 0 \end{cases}$ $\Rightarrow$ We have to compute left and right limits separately.

$$\lim_{\substack{h \to 0^+ \\ h>0}} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

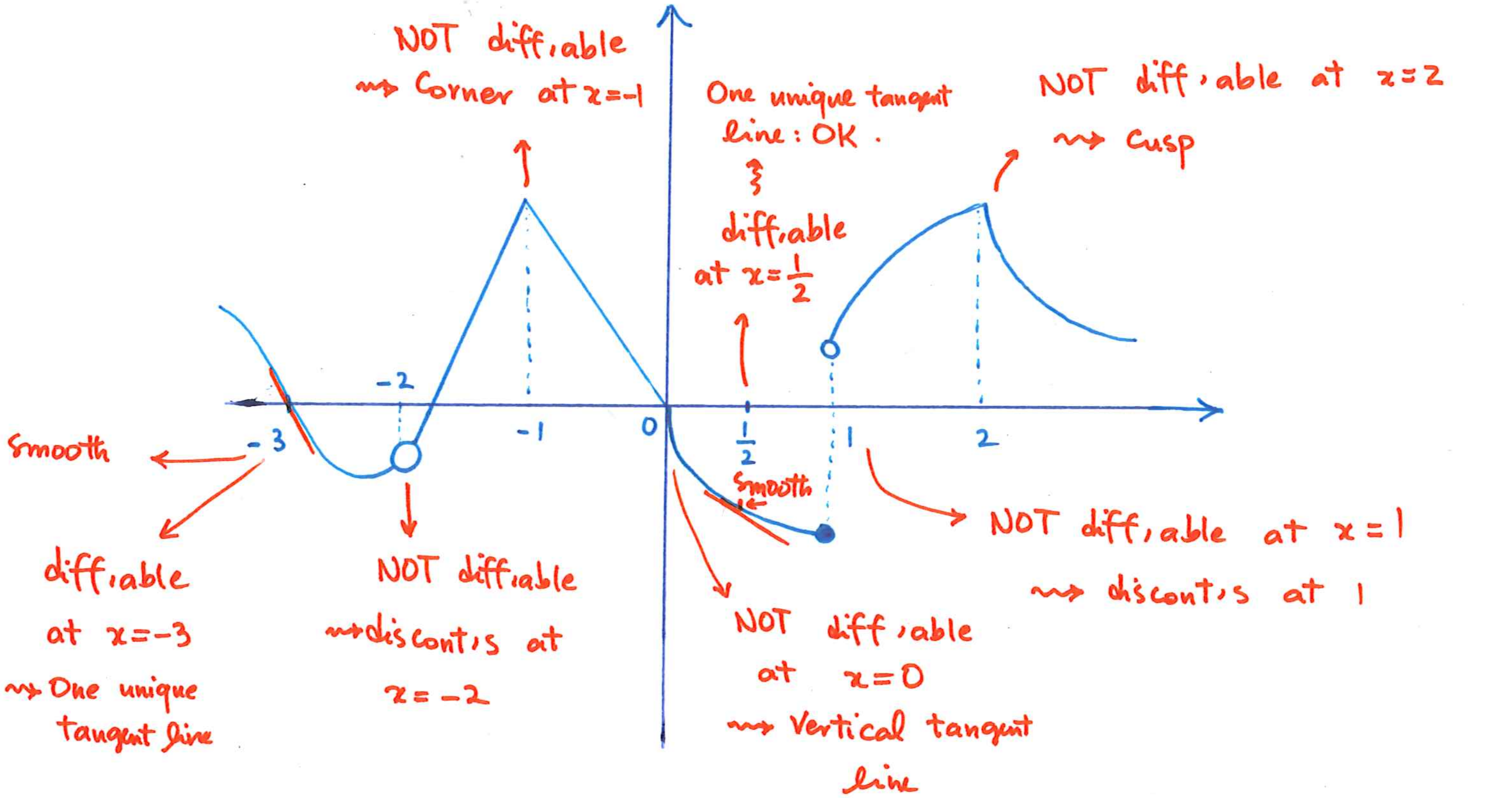
$$\lim_{\substack{h \to 0^- \\ h<0}} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$\Longrightarrow$ The limit DNE

$\Rightarrow$ f is NOT diff'able at $x = 0$

Practice. Determine if f is diff'able at

$x = -3, -2, -1, 0, \frac{1}{2}, 1, 2$



Practice. True / False.

1) f is cont's then f is diff'able. F

Counter example: above function: at x=-1, x=2, x=0

2) f is discont's ⇒ f is NOT diff'able. T

3) f is diff'able ⇒ f must be continuous. T

Exponential Functions.

Exponential functions have the variable x (or any other independent variable) in the exponent.

Compare : x^2 , x^3 , t^4 , a^5 are polynomials.

But 2^x , 3^x , 4^t , 5^a are exponentials.

In general, if "b" is a real number (a constant) then $f(x) = b^x$ is an exponential function.

(b: base; x: exponent/power)

Dumb Cases:

- $b = 0 \rightsquigarrow f(x) = 0^x = 0$
- $b = 1 \rightsquigarrow f(x) = 1^x = 1$

0^0 is an special case!

We also consider the cases where the base "b" is positive therefore:

Exponential Function ← $f(x) = b^x \quad b \neq 0,\ b \neq 1,\ b > 0$

Laws of Exponents (Review)

Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = \frac{1}{x}$	$4^{-1} = \frac{1}{4}$
$x^m \cdot x^n = x^{m+n}$	$x^2 \cdot x^3 = x^{2+3} = x^5$
$x^m / x^n = x^{m-n}$	$x^6 / x^2 = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2 \cdot 3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$
$x^{-n} = \frac{1}{x^n}$	$x^{-3} = \frac{1}{x^3}$
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$x^{2/3} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2$

Example.

$f(x) = 2^x$, Sketch the graph:

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$

⊛ Outputs are all **positive**.

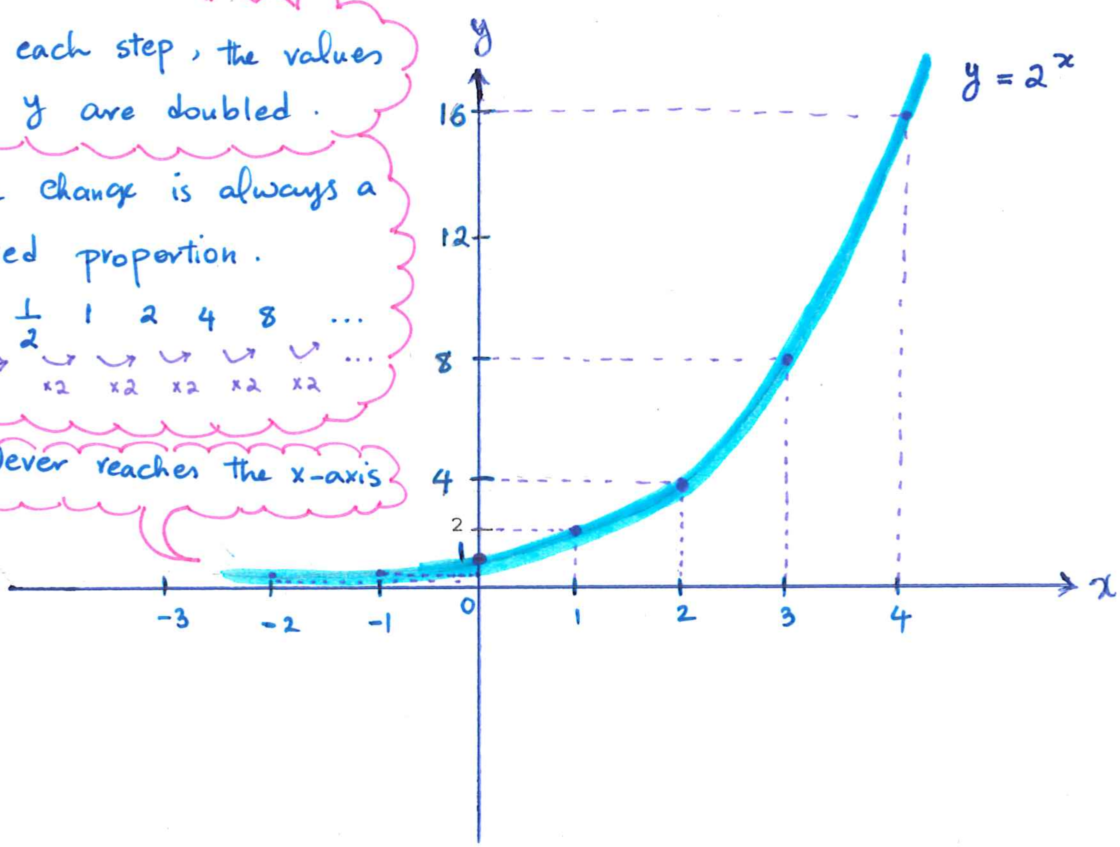
⊛ At each step, the values of y are doubled.

⊛ The change is always a fixed proportion.

$\frac{1}{4} \quad \frac{1}{2} \quad 1 \quad 2 \quad 4 \quad 8 \quad \ldots$

$\times 2 \quad \times 2 \quad \times 2 \quad \times 2 \quad \times 2 \quad \times 2 \quad \ldots$

⊛ Never reaches the x-axis



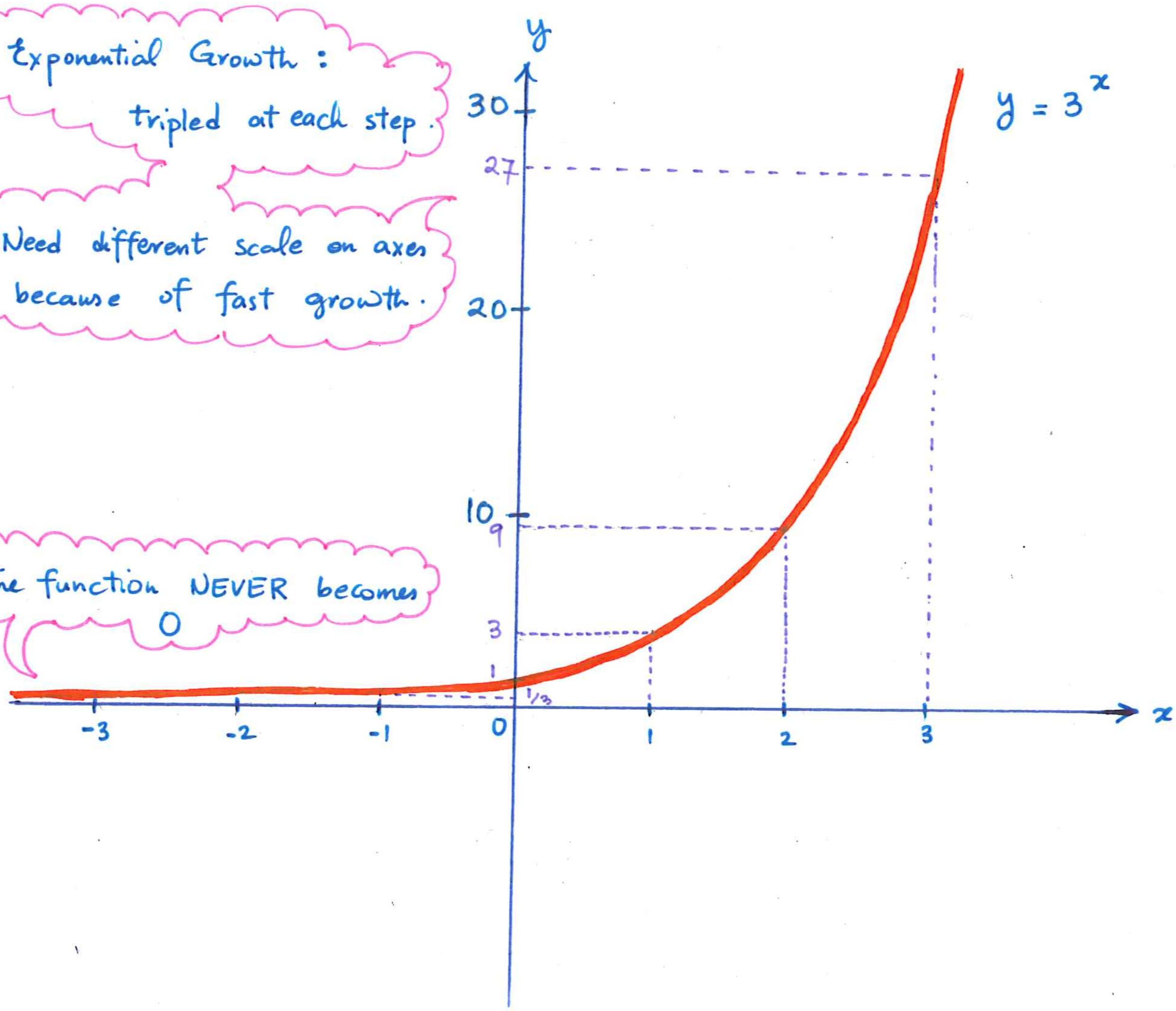
Example. $f(x) = 3^x$, sketch the graph.

x	$y = 3^x$
-3	$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$
-2	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$
3	$3^3 = 27$

* Exponential Growth: tripled at each step.

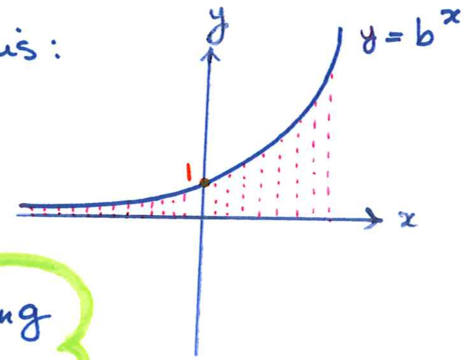
* Need different scale on axes because of fast growth.

* The function NEVER becomes 0



Observations : $f(x) = b^x$

- Exponential Functions look like this:



They start small – very small and very close to "$y = 0$" – and then once they start growing they grow faster and faster.

"Exponential Growth" is referred to this behaviour.

- $f(0) = b^0 = 1 \Rightarrow (0,1)$ is always on the function

$\Rightarrow$ **y-int = 1**

- Exponential functions never reach the x-axis.

$\Rightarrow f(x) \neq 0$

$\Rightarrow$ **NO x-int**

- Exponential functions are always positive. (above y-axis)

$\Rightarrow f(x) > 0$

- Therefore:

Domain = $(-\infty, +\infty)$

Range = $(0, +\infty)$

0 NOT included!

Very Important exponential function :

$f(x) = e^x$ ⇝ Natural exponential function.

This function appears in many natural phenomena, and in general we just call it the exponential function.

⇝ What is "e" ? $2 < e < 3$

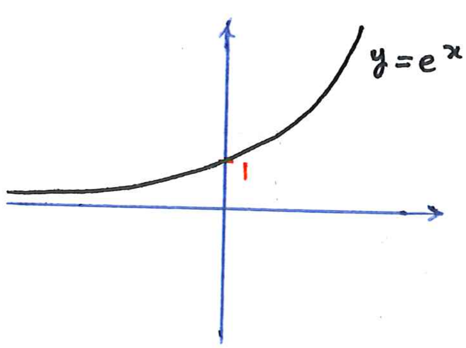
"e" is a math constant, and it is approximately

$$e = 2.718281828459 05 \ldots$$

like π ; e is also an irrational number, and there are different ways to define this number. We'll see one of them later.

⊛ One of the important applications of $y = e^x$ is in the study of "compound interest".

⊛ All the properties of $y = b^x$ applies to $y = e^x$.



✓ Domain = $(-\infty, \infty)$

✓ Range = $(0, +\infty)$ ⇝ Always positive

✓ y-int = 1

✓ NO x-int ⇝ Does NOT cross the x-axis.

Solving Equations with exponentials.

To solve exponential equations in this part we will NOT use logarithms, so we need to make the two sides of the "equal" sign comparable. The terms on both sides must have the same base, so we set the powers equal to each other.

Examples. Solve the following equations.

1) $3^x = 9$ ⇝ must have the same "base"

⇝ $9 = 3^2$

$\Rightarrow 3^x = 3^2 \Rightarrow$ $\boxed{x = 2}$

2) $10^{1-x} = 1000$

$\Rightarrow 10^{1-x} = 10^3 \Rightarrow 1 - x = 3$

$\Rightarrow$ $\boxed{x = 1 - 3 = -2}$

3) $e^x = 1$

$\Rightarrow e^x = e^0 \Rightarrow$ $\boxed{x = 0}$

4) $x - xe^{2x+1} = 0$

Factor x

$\Rightarrow x(1 - e^{2x+1}) = 0 \Rightarrow$ $\boxed{x = 0}$ One solution

or $1 - e^{2x+1} = 0$

$$1 - e^{2x+1} = 0 \Rightarrow e^{2x+1} = 1 = e^0$$

$$\Rightarrow 2x + 1 = 0$$

$$\Rightarrow 2x = -1$$

$$\Rightarrow \boxed{x = -\frac{1}{2}}$$ Another solution

5) $e^x + (x+1)e^x = 0$

Factor e^x $\Longrightarrow$ $e^x(1 + (x+1)) = 0$

$$\Rightarrow e^x(x+2) = 0$$

$$\Rightarrow e^x = 0 \rightsquigarrow \text{NOT possible}$$

(e^x never becomes 0, always positive)

or

$$x + 2 = 0 \Rightarrow \boxed{x = -2}$$ the only solution