Geometrical Problems

(Derivative & Tangent lines)

Lecture 12

Oct 18

Recall that in Lecture 10, we stressed that the three following concepts are all equivalent :

Instantaneous Rate of Change at $x = a$ $\Longleftrightarrow$ $f'(a)$ $\Longleftrightarrow$ Slope of the tangent line at $x = a$: m_{tan}

And they are all equal to

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (*)$$

Now that we know the power rule :

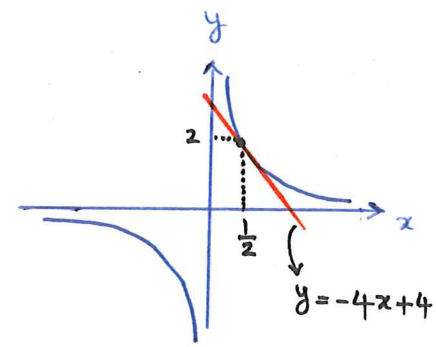
$$\left(x^{●}\right)' = ● \, x^{● - 1}$$

We can use this shortcut to avoid the computation of limit $(*)$.

Go back to the example in a previous lecture, we redo the question, this time we use power-rule.

$\boxed{\text{ADVICE}}$ In geometrical problems, it is usually helpful to roughly sketch the graph and the situation given in the question.

Example. Find the equation of the tangent line to the graph of $y = \frac{1}{x}$ at $x = \frac{1}{2}$?



tangency point : $x = \frac{1}{2}$

$y = \frac{1}{\frac{1}{2}} = 2$ $\Rightarrow (\frac{1}{2}, 2)$ ✓

$m_{tan} = f'(\frac{1}{2})$

$f(x) = \frac{1}{x} = x^{-1} \xRightarrow[\text{Rule}]{\text{Power}} f'(x) = -1\, x^{-1-1}$

$= -x^{-2}$

$= -\frac{1}{x^2}$

$\Rightarrow f'(\frac{1}{2}) = -\frac{1}{(\frac{1}{2})^2} = -\frac{1}{\frac{1}{4}} = -4$

$\Rightarrow m_{tan} = -4$ ✓

This is what we found earlier from limit formula.

Equation : $y - f(a) = f'(a)(x-a)$

$y - f(\frac{1}{2}) = f'(\frac{1}{2})(x - \frac{1}{2})$

$\Rightarrow y - 2 = -4(x - \frac{1}{2})$

$\Rightarrow \boxed{y = -4x + 4}$

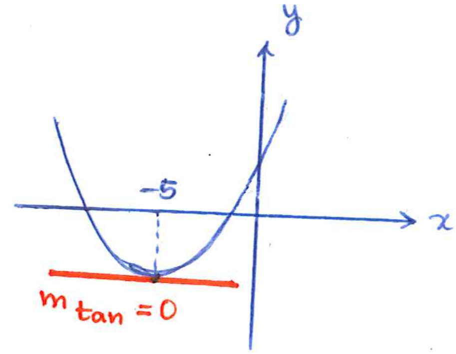
Now, let's do some more complicated examples.

Example 1.

Find "a" such that the function

$$f(x) = 2x^2 + 5ax + 1$$

has a horizontal tangent line at $x = -5$.



Horizontal tangent line $= m_{tan}$ at $x=-5$ $= 0$

And $m_{tan} = f'(-5)$

So we need to find "a" such that $f'(-5) = 0$

By Power rule : $f'(x) = 2\cdot(2x) + 5a = 4x + 5a$

$\Rightarrow \quad f'(-5) = 4\cdot(-5) + 5a = 0$

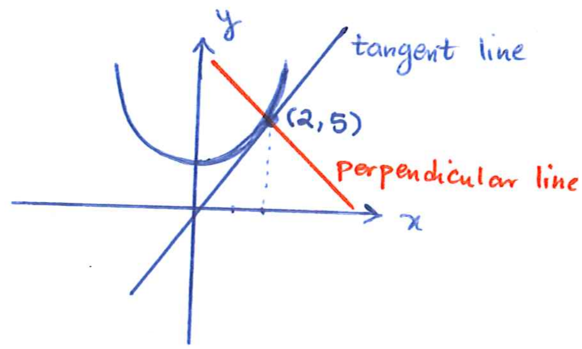
Solve for "a" $\quad -20 + 5a = 0$

$\Rightarrow \quad -20 = -5a$

$\Rightarrow \quad \boxed{a = 4}$

Example 2.

Find the equation of the line perpendicular to the graph of $y = x^2 + 1$ at $(2, 5)$.



tangent line $\perp$ perpendicular line

$$\Rightarrow \quad \boxed{m_{tan} = -\frac{1}{m_{per}}}$$

We know at $x = 2$: $m_{tan} = f'(2)$

Now apply power rule to $f(x) = x^2 + 1$

we have $f'(x) = 2x$

$$\Rightarrow \quad m_{tan} = f'(2) = 4$$

$$\Rightarrow \quad m_{per} = -\frac{1}{4}$$

We have the slope of per. line and we have a point $(2,5)$ on the line, so the equation:

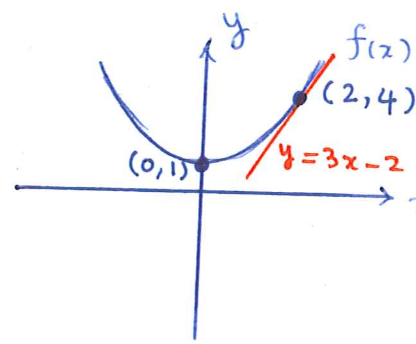
$$y - y_1 = m_{per}(x - x_1)$$

$$\Rightarrow \quad y - 5 = -\frac{1}{4}(x - 2)$$

$$\Rightarrow \quad y = -\frac{1}{4}x + \frac{1}{2} + 5 \quad \Rightarrow \quad \boxed{y = -\frac{1}{4}x + \frac{11}{2}}$$

Example 3.

Find values for a, b, c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(0,1)$ and is tangent to the line $y = 3x - 2$ at the point $(2,4)$.



Information that we have :

I • $(0,1)$ on the graph $\rightsquigarrow f(0) = 1$

II • $(2,4)$ tangency point $\rightsquigarrow f(2) = 4$

III • $y = 3x - 2$ is tangent at $(2,4)$ $\rightsquigarrow m_{tan} = 3 = f'(2)$

From I : $f(0) = a.(0)^2 + b.0 + c = 1 \Rightarrow \boxed{c = 1}$

From II : $f(2) = a.(2)^2 + b.2 + 1 = 4 \Rightarrow 4a + 2b + 1 = 4$

From III : $f'(x) = a(2x) + b = 2ax + b$

$f'(2) = 4a + b = 3$

II & III : $\begin{cases} 4a + 2b = 3 \\ 4a + b = 3 \end{cases} \Rightarrow \boxed{b = 0} \rightarrow 4a = 3 \Rightarrow \boxed{a = \frac{3}{4}}$

Example 4

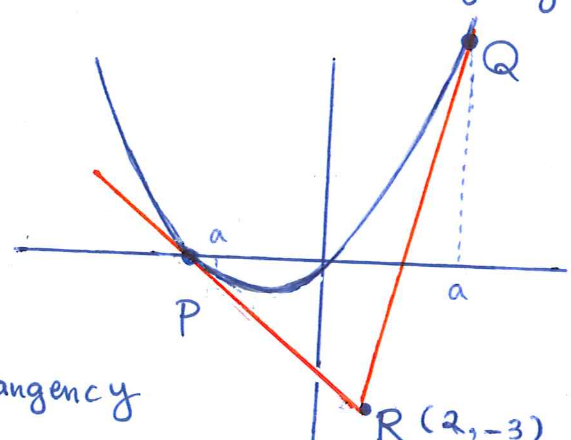
Find the equation of both lines that pass through the point $(2,-3)$ and are tangent to the parabola

$$y = x^2 + x .$$

Q: Why do we have two tangent lines?

Answer: Because $(2,-3)$ is NOT on the graph, in other words $(2,-3)$ is NOT the tangency point.

We want to find the equation of the two red lines.



1st step: Find the two tangency points P and Q

P and Q are on the parabola, so if they have an x-coordinate, say "a", their y-coordinate must be

$f(a) = a^2 + a \quad \Rightarrow \quad (a, a^2 + a)$

Let $R = (2, -3)$, then we know the slope of each tangent line at $x = a$ is $f'(a)$, so $m_{tan} = f'(a)$.

$$f(x) = x^2 + x \Rightarrow f'(x) = 2x + 1$$

$$\Rightarrow f'(a) = 2a + 1 = m_{tan} \qquad (\mathrm{I})$$

Also, slope of each line can be found by the two point formula :

line RQ and PR : slope between $(2, -3)$ and $(a, a^2 + a)$

$$\Rightarrow m_{tan} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{(a^2 + a) - (-3)}{a - 2} \qquad (\mathrm{II})$$

$\rightsquigarrow$ I must be equal to II :

$$\frac{a^2 + a + 3}{a - 2} = 2a + 1$$

$$\Rightarrow a^2 + a + 3 = (a - 2)(2a + 1)$$

$$\Rightarrow a^2 + a + 3 = 2a^2 + a - 4a - 2$$

Simplify

$$\Longrightarrow a^2 - 4a - 5 = 0$$

Factorize

$\Rightarrow$ $$(a-5)(a+1)=0$$

$$a=5 \quad , \quad a=-1$$

these two values of "a" are the x-coordinates for the two tangency points. Q has the positive $x \rightsquigarrow a=5$

P has the negative $x \rightsquigarrow a=-1$

P and Q are on the graph, so to find their y-coordinates we use the equation of the function: $f(x)=x^2+x$

$$a=5 \Rightarrow f(5)=5^2+5=30 \Rightarrow \boxed{Q=(5,30)}$$

$$a=-1 \Rightarrow f(-1)=(-1)^2+(-1)=0 \Rightarrow \boxed{P=(-1,0)}$$

2nd step : Find the slope of each line

$$m_{tan}=f'(a) \underset{(\text{From I})}{=} 2a+1$$

RQ : ✓ tangency point : $(5,30)$

✓ $m_{tan}=f'(5)=2.5+1=11$

✓ Equation : $y-f(a)=f'(a)(x-a)$

$$y-30=11(x-5)$$

$$\Rightarrow y=11x-55+30$$

$$\Rightarrow \boxed{y=11x-25}$$

PR : ✓ tangency point : $(-1, 0)$

✓ $m_{tan} = f'(-1) = 2.(-1) + 1 = -1$

✓ Equation : $y - 0 = -1\,(x-(-1))$

$$\Rightarrow \quad y = -(x+1)$$

$$\Rightarrow \quad \boxed{y = -x - 1}$$