Review Notes Dec 16

Trig review: Refer to the trig sheet I gave in class (also webpage)

Differentiability

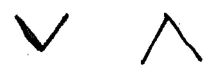
Derivative Definition at point $x=a$: $$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$ → Know this definition of the derivative

- From the limit definition:

  ↪ f is diff'able at "a" if the limit above exists.

- From the graph:

  ↪ f must be continuous

  & f has NO corner 

  & NO cusp 

  & NO vertical tangent line.

f is discontinuous (HoLe or Jump) ⇝ NOT diff'able

f is continuous ⇝ NOT necessarily diff'able

⇒ Continuity is NOT sufficient for differentiability.

- If you are given a piecewise function, take the derivative of each piece separately and check if the derivative evaluated at "a" is equal for each piece.

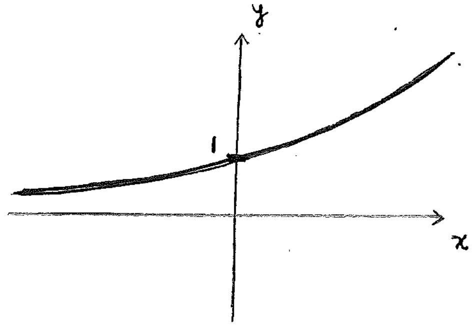
  (Do NOT forget that if the function is NOT defined at a point, it's derivative is NOT defined either.)

Derivative Definition as a function of x : $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

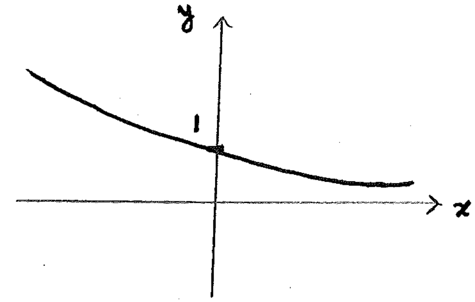
Exponential Functions

$e \approx 2.71$

$f(x) = e^{x}$



$f(x) = e^{-x}$



- e^{x} is always positive
  $\rightsquigarrow$ Never comes below x-axis
  $\rightsquigarrow$ NO x-intercept
- $f(0) = e^{0} = 1 \rightsquigarrow y = 1$ is y-int
- Dom $= (-\infty, \infty)$, Range $= (0, +\infty)$
- $f'(x) = e^{x} > 0 \rightsquigarrow$ Always INCREASING

- e^{-x} Always positive
  $\rightsquigarrow$ Never comes below x-axis
  $\rightsquigarrow$ NO x-int
- $f(0) = e^{-0} = 1 \rightsquigarrow y$-int : 1
- Dom $= (-\infty, \infty)$, Range $= (0, +\infty)$
- $f'(x) = -e^{-x} < 0 \rightsquigarrow$ Always Decreasing

⊛ Recall from the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^{x}}{h} = e^{x} \Rightarrow \boxed{f'(0) = \lim_{h \to 0} \frac{e^{h} - 1}{h} = 1}$$

One way to define "e":
A number such that
$$\lim_{h \to 0} \frac{e^{h} - 1}{h} = 1$$

⊛ By chain Rule:

$$f(x) = e^{▨} \Rightarrow f'(x) = \underbrace{e^{▨}}_{\text{outside derivative}} \cdot \underbrace{▨'}_{\text{inside derivative}}$$

Example.

$$f(x) = e^{x^2 - x} \Rightarrow f'(x) = e^{x^2 - x} \cdot (2x - 1)$$

$$f(x) = e^{\sin x} \Rightarrow f'(x) = e^{\sin x} \cdot \cos x$$

Exponent Laws :

- $e^{x+y} = e^{x} \cdot e^{y}$
- $e^{x-y} = e^{x} \cdot e^{-y} = \dfrac{e^{x}}{e^{y}}$
- $(e^{x})^{y} = e^{xy}$

Example:

$$\frac{e^{(x^3)}\, e^{(2x^2)}\, e^{x}}{e^{3}} = \frac{e^{x^3+2x^2+x}}{e^{3}}$$

$$= e^{x^3+2x^2+x-3}$$

⊛ For any other base : $f(x) = b^{x}$

- $b^{x} \cdot b^{y} = b^{x+y}$
- $b^{x}/b^{y} = b^{x-y}$
- $b^{x} \cdot a^{x} = (ba)^{x}$
- $b^{x}/a^{x} = \left(\frac{b}{a}\right)^{x}$

- For any power or exponential functions when you move terms over the fraction the sign of the exponent changes.

$$x^{-2} = \frac{1}{x^2} \quad , \quad \frac{1}{x^5} = x^{-5} \quad , \quad \frac{1}{2^x} = 2^{-x}$$

- Fractional exponents are written in the form of root functions.

inside ← power of x

$$x^{\frac{2}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2} \quad , \quad x^{-\frac{1}{3}} = \frac{1}{x^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{x}}$$

outside ← order of root

⊛ Solving exp equations : Do NOT forget that we can NOT have

$$e^{\square} = 0 \quad \text{or} \quad e^{\square} = \text{Negative number}$$

⇒ NO solution for these cases.

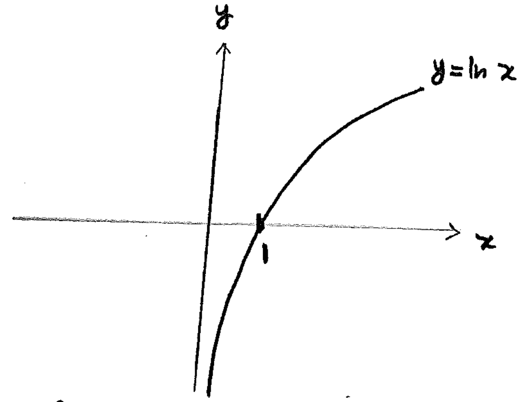
Functions closely related to exponentials ⇝ Logarithmic Function:

base $= e$ ⇝ $y = e^{x}$ $\xrightarrow{\text{inverse}}$ $y = \log_e x = \ln x$

Natural logarithm $y = \ln x$ is the inverse of $y = e^{x}$, so each property for $y = e^{x}$ can be swapped to find the property for $y = \ln x$.

$y = \ln x$

Graph: Reflect e^x over the line $y = x$



x-int $\rightsquigarrow$ $y=0$ $\rightsquigarrow$ $\ln x = 0$ $\rightsquigarrow$ $x = 1$

y-int $\rightsquigarrow$ $x = 0$ $\rightsquigarrow$ $\ln 0$ = NOT defined $\rightsquigarrow$ NO y-int

Dom $= (0, +\infty)$ $\rightsquigarrow$ Only positive numbers can sit for x.

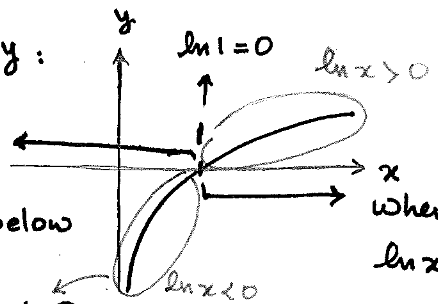
$\ln 0$ X $\quad \ln(-1)$ X $\quad \ln -e^x$ X

Range $= (-\infty, +\infty)$

$$\boxed{y' = \frac{1}{x}} \longrightarrow \text{Rule} \xrightarrow{\text{Chain}} \text{Rule} \qquad y = \ln \square$$

$$\Rightarrow \quad y' = \underbrace{\frac{1}{\square}}_{\text{outside for "ln"}} \cdot \underbrace{\square'}_{\text{inside derivative}}$$

Look closely:



when $0 < x < 1$ $\ln x$ is below x-axis $\rightsquigarrow \ln x < 0$

when $x > 1$ $\ln x$ is above x-axis $\rightsquigarrow \ln x > 0$

Example

$\ln \frac{1}{2} \approx -0.69$

$\ln \frac{1}{5} \approx -1.61$

$\ln 0.6 \approx -0.51$

$\ln 2 \approx +0.69$

$\ln 8 \approx +2.07$

$\ln 1.5 \approx +0.41$

Ln laws:

- $\ln 1 = 0$
- $\ln e = 1$
- $\ln xy = \ln x + \ln y$ — Single ln with $\cdot$, breaks to Two ln's with $+$
- $\ln \frac{x}{y} = \ln x - \ln y$ — Single ln with $\div$, breaks into two ln's with $-$
- $\ln x^n = n \ln x$

Ln & e $\rightarrow$ when they sit exactly next to each other, get cancelled.

$$\ln e^{\square} = \square \quad \& \quad e^{\ln \square} = \square$$

*We use this property to simplify terms to solve equations and find derivatives.

Derivative Rules

Operations between functions.

- Sum $\quad y = f(x) + g(x) \rightsquigarrow \boxed{y' = f'(x) + g'(x)}$
- Difference $\quad y = f(x) - g(x) \rightsquigarrow \boxed{y' = f'(x) - g'(x)}$
- Product $\quad y = f(x) \cdot g(x) \rightsquigarrow \boxed{y' = f'(x)g(x) + f(x)\,g'(x)}$ → product rule

$$y = f(x) \cdot g(x) \cdot h(x) \rightsquigarrow y' = f'(x)\,g(x)\,h(x) + f(x)\,g'(x)\,h(x) + f(x)\,g(x)\,h'(x)$$

- Quotient $\quad y = \dfrac{f(x)}{g(x)} \rightsquigarrow \boxed{y' = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}}$ → Quotient rule → Distribute $\ominus$ in all the terms after that.
- Composition $\quad y = f \circ g(x) = \underbrace{f}_{\text{outside}}(\underbrace{g(x)}_{\text{inside}}) \longrightarrow \boxed{y' = \underbrace{f'(g(x))}_{\text{Derivative of outside evaluated at inside}} \cdot \underbrace{g'(x)}_{\text{Derivative of inside}}}$ → chain rule

⊛ Sometimes it's easier to rewrite the quotient $\dfrac{f(x)}{g(x)} = f(x)(g(x))^{-1}$ and apply product rule with chain rule.

especially if the numerator f is a constant number.

Example.

$$y = \frac{10}{x^3+1} = 10(x^3+1)^{-1} = 10 \cancel{\quad}^{-1}$$

$$\Rightarrow y' = 10 \cdot (-1)\,(x^3+1)^{-1-1} \cdot 3x^2$$

$$= -10\,(x^3+1)^{-2} \cdot 3x^2 = \frac{-30x^2}{(x^3+1)^2}$$

⊛ In the computations, when you get a negative power, you don't need to change it to positive, BUT if you are asked to evaluate a function with negative power or find it's domain or other operations it's safer to change it to positive power in the denominator.

Types of functions

- powers $\boxed{y = x^n \rightsquigarrow y' = n x^{n-1}}$ BUT $\boxed{\underbrace{y = c}_{\text{just a number}} \rightsquigarrow y' = 0}$

- constant multiple $y = \underset{\text{number}}{c}\, f(x) \rightsquigarrow y' = \underset{\text{just keep the constant that is multiplied}}{c}\, f'(x)$

- Root functions can be written as power functions.

$$\sqrt{x} = x^{\frac{1}{2}}, \quad \sqrt[3]{x} = x^{\frac{1}{3}}, \quad \frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}, \quad x\sqrt[5]{x} = x \cdot x^{\frac{1}{5}} = x^{1+\frac{1}{5}} = x^{\frac{6}{5}}$$

- Exponentials :

$$\boxed{\begin{array}{ll} y = e^x \rightsquigarrow & y' = e^x \\ \text{For any base}: y = b^x \rightsquigarrow & y' = \ln b \cdot b^x \end{array}}$$

- Logarithm (Natural) : $\boxed{y = \ln x \rightsquigarrow y' = \frac{1}{x}}$

- Trigs :

$$\boxed{\begin{array}{ll} y = \sin x \rightsquigarrow & y' = \cos x \\ y = \cos x \rightsquigarrow & y' = -\sin x \\ y = \tan x \rightsquigarrow & y' = 1 + \tan^2 x = \sec^2 x \end{array}}$$

By quotient rule you can find the derivative of :

$y = \cot x = \frac{\cos x}{\sin x}$

$y = \sec x = \frac{1}{\cos x}$

$y = \operatorname{cosec} x = \frac{1}{\sin x}$

⊛ All these functions can be composed with other functions, and we then need to use chain rule to differentiate.

Ex.

- $y = (2x^3 + \sqrt{x} + 3)^{100} = (2x^3 + x^{\frac{1}{2}} + 3)^{100} = \bigcirc^{100}$

$$\rightsquigarrow y' = 100(2x^3 + x^{\frac{1}{2}} + 3)^{99} \cdot (6x^2 + \tfrac{1}{2} x^{-\frac{1}{2}})$$

- $y = \ln(\sin(x^3)) = \ln \bigcirc \rightsquigarrow y' = \underbrace{\frac{1}{\sin(x^3)}}_{\text{for } \ln} \cdot \underbrace{\cos(x^3)}_{\text{for } \sin} \cdot \underbrace{3x^2}_{\text{for } x^3}$

- $y = \sqrt{\tan x} + 5 e^{-4x} \sin^3 x$

$\sqrt{\tan x} = (\tan x)^{\frac{1}{2}}$, $\sin^3 x = (\sin x)^3$

Keep the ...

product rule

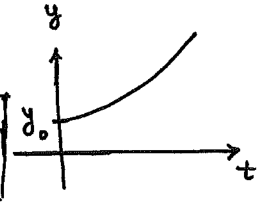
$$\rightsquigarrow y' = \underbrace{\tfrac{1}{2}(\tan x)^{-\frac{1}{2}}}_{\text{for } \bigcirc^{\frac{1}{2}}} \cdot \underbrace{(1 + \tan^2 x)}_{\text{for } \tan}$$

$$+ 5\Big(\underbrace{-4}_{\text{for}}\underbrace{e^{-4x}}_{\text{for}} \sin^3 x + e^{-4x} \underbrace{\cos x}_{\text{for } \sin} \cdot \underbrace{3(\sin x)^2}_{\text{for } \bigcirc^3}\Big)$$

Exponential Models

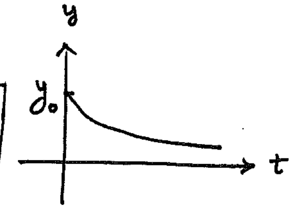
ALWAYS Increasing — Growth

$$\boxed{y(t) = y_0 e^{kt}}$$



ALWAYS decreasing — Decay

$$\boxed{y(t) = y_0 e^{-kt}}$$



are constant numbers ← { y_0 = initial quantity; k: Growth rate ⇝ ALWAYS positive (Rate constant) }

t : time ⇝ independent variable

Growth Rate ← $\boxed{\begin{array}{l} y'(t) = y_0 e^{kt} \cdot k \\ y'(t) = k\, y(t) \end{array}}$

$\boxed{\begin{array}{l} y'(t) = y_0 e^{-kt} \cdot -k \\ y'(t) = -k\, y(t) \end{array}}$ → Decay Rate

Key Feature

at time t : The growth/decay rate is proportional to the amounth of quantity at time t.

Doubling time (100% increase): $\boxed{T_2 = \dfrac{\ln 2}{k}}$ ⟷ Similarly ⟷ Half-life (50% decrease): $\boxed{T_{\frac{1}{2}} = \dfrac{\ln 2}{k}}$

Tripling time: $\boxed{T_3 = \dfrac{\ln 3}{k} = T_{\frac{1}{3}}}$

Second derivative: y'', f'', $\dfrac{d^2f}{dx^2}$, $\dfrac{d^2y}{dx^2}$: The derivative of derivative

Applications :

Height / Distance ⇝ Velocity ⇝ Acceleration

$h(t)$ ⇝ $\underbrace{h'(t)}_{v(t)}$ ⇝ $\underbrace{h''(t)}_{a(t)}$

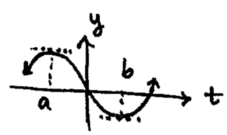
A quantity grows/decays ⇝ Rate of Change in the quantity $\boxed{\text{Growth / Decay Rate}}$ ⇝ Rate of Change in the growth/decay rate

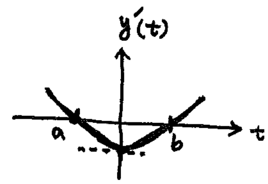
$y(t)$ $\qquad$ $y'(t)$ $\qquad$ $(y')'(t) = y''(t)$

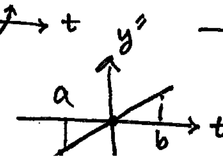
Graphically :

slope at $y(t) = 0$ ⇝ $y'(t)$ is 0 (Horizontal)

slope at $y'(t) = 0$ ⇝ $y''(t) = 0$







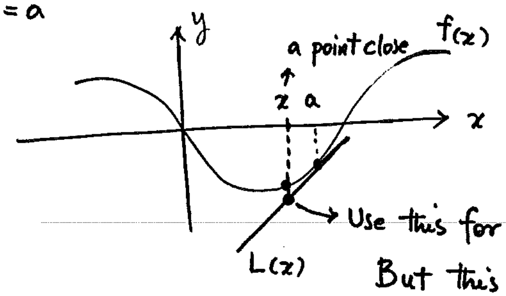
Linear Approximation (linearization)

↓

tangent line to the graph of a function $f(x)$ at a given point $x=a$ and use this line to evaluate at some other points close to "a", instead of using $f(x)$.

find $f(x)$ & $x=a$ ⇝ Find the tangent line: $m_{tan} = f'(a)$, point: $(a, f(a))$: $L(x) = f'(a)(x-a) + f(a)$ ⇝ Another point usually with decimals, plug that into $L(x)$ and conclude $L(x) \approx f(x)$

⊛ If you are asked to graph a function and its linearization, you just should find the $\underbrace{\text{touch point}}_{x=a}$ and sketch the tangent line at the touch point.



Use this for approximation

But this is NOT the touch point of the tangent line.

⊛ Sometimes the function $f(x)$ and the point "a" are not explicitely given and you must find them from the given approximation.

Example : $\ln(1.01)$ ⇝ $f(x) = \ln x$, $a = 1$, $x = 1.01$

$\sqrt{9.99}$ ⇝ $f(x) = \sqrt{x}$, $a = 9$, $x = 9.99$