Functions, Limits & Continuity

Go to the midterm review for a complete review on these topics.

Just a quick recall:

For the $\lim\limits_{x \to a} f(x)$ exist we just need $\boxed{\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)}$

And we <u>don't care</u> about $f(a)$. (It can be a hole, jump, asymptot)

But to have a continuous function, then <u>f(a) matters</u> and we must have

$$\boxed{\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)}$$

Now that we've learned new types of functions: trigs, exp, log, ... all the topics before the midterm can be asked about these new functions: Domain & Range, Solve equations, Finding the limits, Check the continuity and apply IVT to new functions NOT just polynomials:

① Find the following limits:

a • $\lim\limits_{x \to -2} \dfrac{x^2 - 2x - 8}{x + 2} =$

b • $\lim\limits_{x \to -4} \dfrac{x^3 + 64}{x + 4} =$

c • $\lim\limits_{x \to 5^-} \dfrac{2}{(x-5)^3} =$

d • $\lim\limits_{x \to -7^+} \dfrac{-x+1}{x^2(x+7)} =$

e • $\lim\limits_{\theta \to \pi/4} \theta \tan\theta$

f • $\lim\limits_{x \to 2\pi} \left[\sin\left(\frac{x}{3}\right)\right]^2$

g • $\lim\limits_{x \to 0} \dfrac{\cos x + \sin 2x}{x^2 + 3}$

h • $\lim\limits_{x \to \pi} \dfrac{\sin x}{\sec x}$

i • $\lim\limits_{x \to \pi^+} \dfrac{\sec x}{\sin x}$, $\lim\limits_{x \to \pi^-} \dfrac{\sec x}{\sin x}$

j • $\lim\limits_{x \to 2\pi} e^{2\sin 2x} \cdot \cos\dfrac{x}{2}$

K • $\lim\limits_{x \to \pi/2} \sin(x + \cos x) + e^{2x}$

l. $\lim_{x \to 6^-} \ln(6-x) =$ what is the domain of $\ln(6-x)$?

m. $\lim_{x \to 1^-} e^x + \frac{1}{\ln x}$, $\lim_{x \to 1^+} e^x + \frac{1}{\ln x}$

(2) Verify the points at which the following functions are dis cont's ?

a. $f(x) = \begin{cases} 2\cos x + \sin 3x & x < 0 \\ 7x^2 + 2x + \tan x + 2 & x \geq 0 \end{cases}$

e. $f(x) = \frac{x + \cos x}{\sin x}$ in $(0, 2\pi)$

b. $f(x) = \sqrt{\sin x}$

f. $g(x) = \begin{cases} e^{2\sin 2x} & x \leq \pi \\ \ln(\sin(x - \pi/2)) & x > \pi \end{cases}$

c. $g(x) = \ln(\ln x)$

g. $h(x) = \frac{1}{\sqrt{x} - 1} + \frac{1}{\ln x}$

d. $h(x) = \sqrt{\ln x}$

(3) Sketch the following functions.

a. $f(x) = \begin{cases} \sin x & 0 \leq x < \pi \\ x - \pi & \pi \leq x < 2\pi \\ 2 & x \geq 2\pi \end{cases}$

→ What is f's domain and range ?
→ At what points f is NOT differentiable ?

b. $g(x) = \begin{cases} e^x & x < 0 \\ 0 & x = 0 \\ e^{-x} & x > 0 \end{cases}$

→ What are the x and y intercepts of g ?

c. $h(x) = \begin{cases} \sqrt{x} & 0 < x \leq 1 \\ \ln x & x > 1 \end{cases}$

→ Is $h(x)$ invertible ? Why ?

d. $P(x) = \begin{cases} x^2 - \pi x + \frac{\pi^2}{4} & x > \frac{\pi}{2} \\ \cos x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ x^2 + \pi x + \frac{\pi^2}{4} & x < -\pi/2 \end{cases}$

→ What is the domain & range of $P(x)$ = ?
→ What are the points of discontinuity ?
→ What are the points of non-differentiability ?

(3) Assume $\lim_{x \to 0} \frac{\sin x}{x} = 1$, find the following limits:

a. $\lim_{x \to 0} \frac{\sin x}{3x}$ b. $\lim_{x \to 0} \frac{\tan x}{5x}$ c. $\lim_{\theta \to 0} \frac{\pi \cos\theta \sin\theta + \theta}{\theta}$

(4) $\boxed{\text{IVT}}$

a. Show that there is at least one point on the curve $y = x^4 + 2x^2 - 4x - 1$ where the tangent line to the curve is horizontal.

b. Prove that the equation $e^{-x} + 2 = x$ has at least one solution.

c. Prove that there is a real number c such that $\sin c = c$.

d. Show that there is some u such that $0 < u < 2$ and $u^2 + \cos(\pi u) = 4$.

e. Show that the function $f(x) = \ln x \sin x$ has at least one x-intercept on the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

f. Determine if there exists a point at which the slope of the tangent line of $y = xe^{-x^2}$ is $\frac{1}{2}$. (Justify your answer.)

(5) $f(x) = \sqrt{x} - 1$ and $g(x) = f \circ f(x)$.

a) Find a formula for $g(x)$.

b) What is the domain of $g(x)$?

(6) Find the x-int of a) $f(x) = x^2 \sin 2x$ b) $f(x) = (x+1)(2x-1)e^{-x}$

c) $f(x) = (\cos x) \ln x$

(7) Solve the following equations:

a. $3^{x^2 - 3x} = 81$

b. $2e^x + 5 = 115$

c. $64^{7x-8} = 1$

d. $\ln(x+1) = 2$

e. $e^x = 80$

f. $5(8e^{2x} - 3)^3 = 625$

Sometimes the limit definition of the derivative is given and you must find the function f and the point "a" from the definition:

Example. $\lim\limits_{h\to 0} \dfrac{\sqrt{2+h}-\sqrt{2}}{h}$ what is $f(x)$ and what is "a" ?

Compare this with $\lim\limits_{h\to 0} \dfrac{\overbrace{f(a+h)}^{\sqrt{2+h}} - \overbrace{f(a)}^{\sqrt{2}}}{h} \Rightarrow a=2$ and $f(x)=\sqrt{x}$

test : $f(2+h)=\sqrt{2+h}$
$f(a)=f(2)=\sqrt{2}$

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Derivative Definition

- Use the definition of derivative to find the following limits:

1) $\lim\limits_{h\to 0} \dfrac{\tan(3x+3h)-\tan 3x}{h}$

a) 0 b) $3\sec^2(3x)$ c) $\sec^2(3x)$ d) $3\tan^2 3x$ e) DNE

2) $\lim\limits_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$

a) 0 b) 1 c) $\sin x$ d) $\cos x$ e) DNE

3) $\lim\limits_{h\to 0} \dfrac{8(\frac{1}{2}+h)^8 - 8(\frac{1}{2})^8}{h}$

a) 0 b) $\frac{1}{2}$ c) 64 d) $\frac{1}{64}$ e) DNE

4) $\lim\limits_{h\to 0} \dfrac{\ln(e^2+h)-2}{h}$, represents derivative of what function ?

a) $f(x)=\ln x$ for $f'(e)$ b) $f(x)=\ln x$ for $f'(e^2)$ c) $f(x)=\ln(x+e^2)$ for $f'(e)$

d) $f(x)=\ln(x+e^2)$ for $f'(1)$ e) $f(x)=\dfrac{\ln x}{x}$ for $f'(e^2)$

⑨ True / False

a) If f is cont's at $x = a$, then $\lim_{x \to a^-} f(x)$ is always equal to $f(a)$.

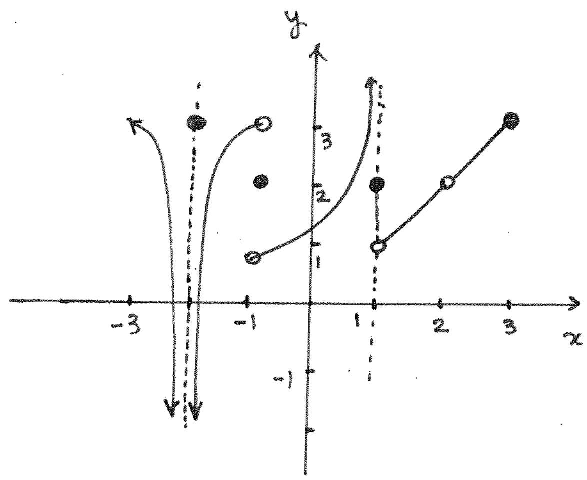
b) If f is NOT cont's at $x = a$, then $\lim_{x \to a^+} f(x)$ cannot exist.

c) If $f(a)$ is NOT defined, then $\lim_{x \to a} f(x)$ Does NOT Exist.

d) If f is NOT differentiable at $x = a$, then f must be discont's at $x = a$.

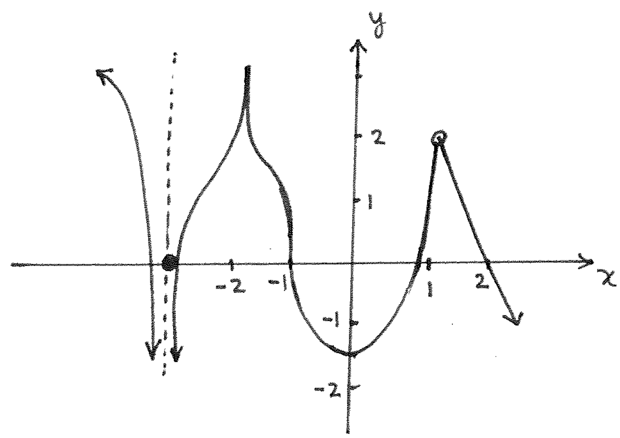
e) If f is discont's at $x = a$, then $f'(a)$ cannot exist.

⑩ For the graph below, determine the continuity & differentiability at each given point and state the reason for your choice.



At

$x = -2, -1, 0, 1, 2, 3$



At $x = -3, -2, -1, 0, 1, 2$

⑪ Use the definition of the derivative to find $f'(1)$ when $f(x) = \dfrac{x^2}{x^2+1}$.

(12) Differentiate the following functions.

a. $f(x) = \dfrac{1}{1+\frac{1}{x}}$

b. $f(x) = \ln(xe^x)$

c. $g(x) = x^3\sqrt[5]{2-x}$

d. $g(x) = \dfrac{\frac{1}{x}+\frac{1}{x^2}}{x-1}$

e. $h(t) = \sqrt{\dfrac{2t+5}{7t-9}}$

f. $h(x) = \tan(\cos x) + (xe^x)^\pi$

g. $a(x) = \ln\left[\left(\dfrac{1-\sin t}{1+\sin t}\right)^2\right]$

h. $r(x) = \dfrac{\ln 5x}{x^5 \ln x^2} + \left(\ln\left(\frac{2}{x}\right)\right)^3$

i. $y(t) = 9x\sin\left(\frac{1}{x}\right)$

j. $F(x) = \ln\left[\dfrac{(2t+1)^2}{(3t-1)^4}\right]$

k. $y = e^{4\tan(\sqrt{x})}$

l. $f(x) = \sqrt{4+\sqrt{5x}}$

(13) Find the derivative at the indicated point

a. $\sin(\sin x)$ at $x=\pi$

b. $\cos x \cdot \log(3x)$ at $x=\pi$

c. $\log\left(\dfrac{x^3}{\sqrt{x}}\right)$ at $x=\frac{5}{2}$

d. $\dfrac{(9-3x)^3}{2x^3}$ at $x=3$

(14) Let $y = e^{ax} + b(x+1)^3$. When $x=0$, suppose that $y' = 0$ and $y'' = 0$. Find the possible values of a and b.

(15) If f and g are differentiable functions such that $f(2)=3$, $f'(2)=-1$, $f'(3)=7$, $g(2)=-5$, $g'(2)=2$. Evaluate the following:

- $(g-f)'(2)$
- $(5f+3g+10)'(2)$
- $(fg)'(2)$
- $(f\circ f)'(2)$
- $\left(\frac{f}{g}\right)'(2)$
- $\left(\frac{f}{f+g}\right)'(2)$

(16)

(a) State the definition of the derivative of $f(x)$ at $x=a$.

(b) Use the definition of derivative to find $f'(2)$ for $f(x) = x + \frac{1}{x}$.

(17) The function $f(x) = \begin{cases} e^x & x \leq 1 \\ mx+b & x>1 \end{cases}$ is differentiable at $x=1$.

Find the values for the constants m and b.

(18) Let $f(x) = 2x + \cos x$. For what values of x is f increasing?

(19) Find equations of all lines that are horizontal and tangent to the curve $y = \dfrac{e^{2x}}{(x-1)^2}$.

(20) Find the equation of the tangent line to the curve

$f(x) = x \ln\left(\frac{1}{x}\right)$ at $x=e$.

(21) Find a linear approximation to the function $f(x) = 3xe^{2x-10}$ at $x=5$.

(22) Use linear approx. to find the following formulas:

$\ln(1+x) \approx x$, $\tan x \approx x$, $e^x \approx 1+x$

(23) Estimate the following values:

$\sqrt[3]{65} =$, $\sin\left(\frac{\pi}{4} + 0.02\right) =$, $\sqrt[4]{16.001} =$

(24) Find the linear approximation of $f(x) = \sqrt{1-x}$ at $x=0$. Use it to approximate $\sqrt{0.9}$.

(25) Find all tangent lines to $y = \ln(1+x^2)$ that are parallel to the line $x - y = 3$

Exp Growth / Decay (26)

a. Assuming the world population grows exponentially and in 1927 it is 2 billion and in 1974 it's about 4 billion.

- Estimate when it reached 6 billion.
- What will the population of the world be in 2100 ?

b. The mass of a sample of Polonium-210, initially 6 grams, decreases at a rate proportional to the mass. After one year, 1 gram remains. What is the half-life ? At what rate the mass decreases initially ?

c. Radium-221 has a half-life of 30 seconds. How long does it take for only 0.01% of an original sample to be left ?

d. A chemical sample has a half-life of 138 days. What percentage of the sample decays in a day ?

e. A particular bactenial culture grows at a rate proportional to the number of bacteria present. If the size of the culture triples every nine hours how long does it take the culture to double ?

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(27) Let $s(t) = t^3 - \frac{21}{2}t^2 + 30t$ be the position function of a particle that is moving in a straight line. What is the acceleration of the particle when the particle has stopped moving ?

(28) An object attached to a spring is at a height $h(t) = 3 - \cos(2t)$ feet above the floor, t seconds after it is released.

a) At what height was it released ?

b) Determine acceleration at time t.

(29) You are moving from left to right along the graph of 2^{-x}. If x-coordinate of your location at time t is $x(t) = 5t + 1$, then how fast is your y-coordinate of your location changing ?