WRITTEN ASSIGNMENT 2.1

Due date: Friday, Jan 12, 2018 (in class)

Hand in full solutions to the questions below. Make sure you justify all your work and include complete arguments and explanations. Your asnwers must be clear and neatly written, as well as legible (no tiny drawings or micro-handwriting please!). Your answers must be stapled, with your name and student number at the top of each page.

1. In this question, you will be reviewing your December exam and reflect on your errors. Solutions to some of the questions from the December exam are posted at

   `http://www.math.ubc.ca/~costanza/m110/ex-dec_sol.pdf`

   (Exams were returned in class, talk to your instructor if you haven't picked up your exam yet).

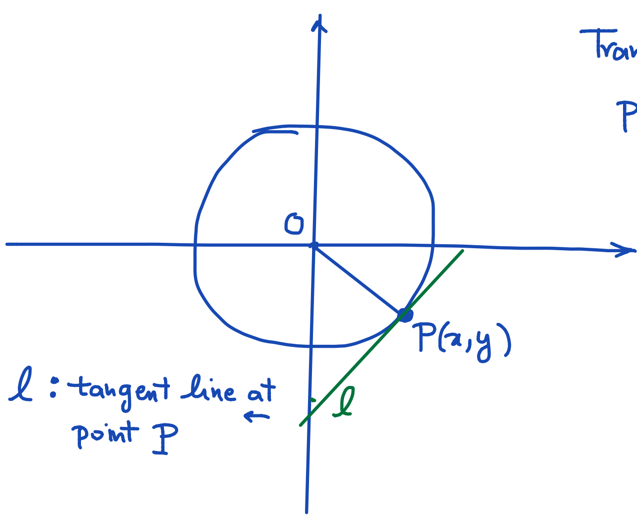
   (a) Review solutions to Question 1 and 4 of the December exam, then compare them with your answers on the exam. If your answers are incorrect, explain why. Make sure you explain what key concept(s)/skill(s) the question was testing and how you would correct your work. Your answers should include explanations and, if needed, some calculations, but there is no need to reproduce a full solution. Imagine the reader is a fellow student in your class who doesn't know how to answer the problem and needs detailed explanations on the concepts and techniques required to tackle the problem in question.

   If you scored at least 85% on either one of these questions, pick two other questions where you made some significant mistakes and comment on your errors.

   (b) Redo question 7 and 9 of the exam. If you need help getting started on these questions, you should look at your past midterm exam (the October exam) or past homework, quizzes or class notes.

2. Here you need to construct a mathematical argument to show, using implicit differentiation, that the tangent line at any point P to a circle with centre at the origin O is perpendicular to the radius OP.

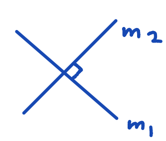
   Hint: The equation of a circle centred at the origin with radius r is $x^2 + y^2 = r^2$. Consider a point $P(x, y)$ on the circle and find the slope of the tangent line to the circle at that point. Then think about the segment OP from the origin to P, what's its slope?



Translate the question: we want to show OP is Perpendicular to the tangent line ℓ.

⟶ What is the relation between perpendicular lines? The slope of one line is negative and reciprocal of the slope of the other

↳ $m_1 = -\frac{1}{m_2}$



⇒ So find slope of ℓ and slope of OP and verify the above relation.

Slope of the tangent line at any point is equal to y' at that point.

We find y': $\quad x^2 + y^2 = r^2 \rightarrow$ radius : constant number

derivative $= 0$

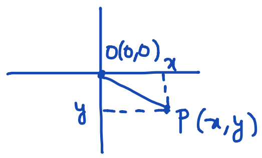
Take the derivative from the both side.

(*) NOT all letters are variables, some are just constants.

$\xrightarrow[\text{diff}]{\text{implicit}} y = y(x)$

$$\xrightarrow{\text{Derive}} 2x + 2y\,y' = 0 \Rightarrow 2yy' = -2x \Rightarrow \boxed{y' = \frac{-x}{y}} \rightarrow m_{tan}$$

Now we want the slope of OP : for any line $m = \dfrac{\text{rise}}{\text{run}}$.



$$m_{OP} = \frac{y-0}{x-0} = \frac{y}{x}$$

Compare the two slopes :

$$m_1 = -\frac{x}{y} \,, \quad m_2 = \frac{y}{x}$$

one is negative reciprocal of the other.

$\Longrightarrow$ Tangent line is perpendicular to OP .

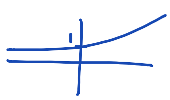
Q7

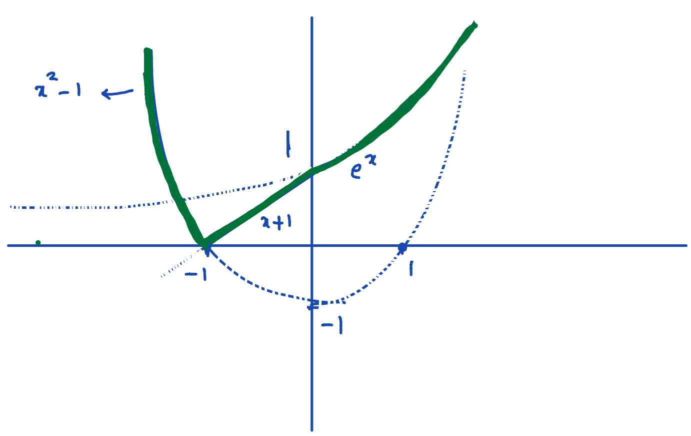
$$h(x) = \begin{cases} x^2 - 1 & x \leq -1 \\ x + 1 & -1 < x < 0 \\ e^x & x \geq 0 \end{cases}$$

$\xrightarrow[\text{Parabola}]{y = x^2 - 1}$ $\begin{cases} y\text{-int} \xrightarrow{x=0} y = -1 \\ x\text{-int} \xrightarrow{y=0} x^2 = 1 \rightarrow x = 1, -1 \end{cases}$

$\xrightarrow[y = x+1]{}$ $\begin{cases} x\text{-int} \xrightarrow{y=0} x = -1 \\ y\text{-int} \xrightarrow{x=0} y = 1 \end{cases}$

$y = e^x$

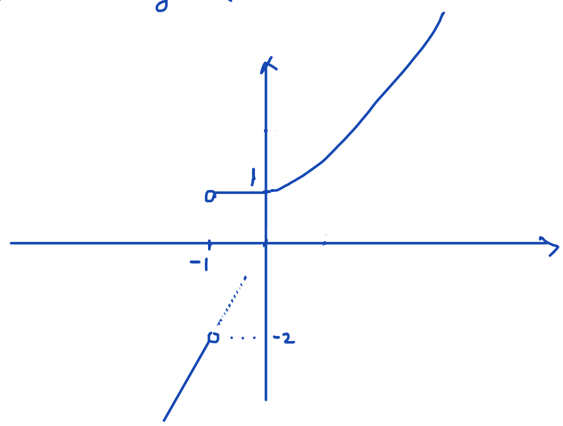




The function is continuous everywhere. However at $x=-1$ its not differentiable it is a cusp, at $x=1$ it is differentiable because the slope of the tangent line from left and right is equal to 1.

$$y=x^2-1 \longrightarrow y'=2x \quad x<-1 \xrightarrow{x=-1} y'=-2 \quad \neq$$
$$y=x+1 \longrightarrow y'=1 \quad -1<x<0$$
$$y=e^x \longrightarrow y'=e^x \quad x\geq 0 \xrightarrow{x=0} y'=1 \quad =$$

$$\Rightarrow y'=\begin{cases} 2x & x<-1 \\ 1 & -1<x<0 \\ e^x & x\geq 0 \end{cases}$$



Q9 $f(x)=\frac{1}{4}x^4+x$
$g(x)=\frac{1}{6}x^6$ $\Rightarrow$ Do they have parallel tangent lines at $x=c$? $\xrightarrow{\text{means}}$ Are the slopes of tangent lines equal at $x=c$?

$\xrightarrow{\text{means}}$ $f'(c)\stackrel{?}{=}g'(c)$

$f'(x)=x^3+1$
$g'(x)=x^5$ $\Rightarrow$ First, We use IVT to show that there is a c in $(0,2)$ such that $f'(c)=g'(c)$ i.e. $c^3+1=c^5$.

We define a new function $h(x)=x^3+1-x^5$. h is continuous everywhere

and $h(0)=0+1-0=1>0$

$h(2)=8+1-32=-23<0$

Since h changes sign in $(0,2)$ by IVT there exists a number c in $(0,2)$ such that $h(c)=0 \xrightarrow[h]{\text{Definition of}} c^3+1-c^5=0 \Rightarrow c^3+1=c^5$

Now we should decide whether c is closer to 0 or 2. For this, we split

The interval $(0,2)$ into two subintervals say $(0,1)$ and $(1,2)$ and again we check signs for each of these intervals:

$h(0) = 1 > 0$

NOT good

$h(1) = 1 + 1 - 1 = 1 > 0$

Good $\rightsquigarrow$ Applying IVT on the smaler interval $(1,2)$ shows that c is closer to 2.

$h(2) = -23 < 0$