Welcome Back to MATH 110, Section 001

Lecture 1, January 3

Review: The Chain Rule : used for composition of function.

$$\text{if} \quad y = \underbrace{f}_{\text{outside}}(\underbrace{g(x)}_{\text{inside}}) \Rightarrow y' = \underbrace{f'(g(x))}_{\substack{\text{take the derivative}\\ \text{of the outside function}\\ \text{and evaluate it at the inside}}} \overset{\text{multiply}}{\cdot} \underbrace{g'(x)}_{\substack{\text{derivative of}\\ \text{the inside}\\ \text{function.}}}$$

Similarly if $y = f(g(h(x)))$ then

$$\Rightarrow \quad y' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Examples: (1) $y = \underbrace{e}_{\text{outside}}^{\underbrace{x^2-2x}_{\text{inside}}}$; find y'.

$$y' = e^{x^2-2x} \cdot (2x-2)$$

(2)
$$y = \underbrace{\cos}_{\text{outside}}(\underbrace{2x+e^x}_{\text{inside}}) + \underset{\text{out}}{\sqrt{\underbrace{x^3-1}_{\text{in}}}} \rightarrow (x^3-1)^{\frac{1}{2}}$$

$$y' = \underbrace{-\sin(2x+e^x)}_{f'(g(x))} \cdot \underbrace{(2+e^x)}_{g'(x)} + \underbrace{\frac{1}{2}(x^3-1)^{\boxed{\frac{1}{2}-1} = -\frac{1}{2}}}_{f'(g(x))} \cdot \underbrace{3x^2}_{g'(x)}$$

$$= -(2+e^x)\sin(2x+e^x) + \frac{1}{2\sqrt{x^3-1}} \cdot 3x^2$$

simplification : It is better not to have negative exponents and rewrite them as ~~a~~ fraction with positive exponent. This helps us see the roots of denominator.

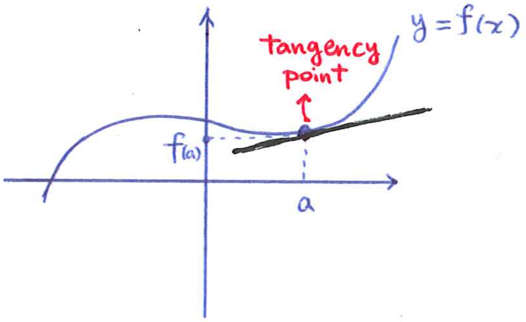
(3) $y = (f(x))^3$ and $f(1) = 2$, $f'(1) = 1$, find $y'(1)$.

$$y = \underset{\text{inside}}{(f(x))}^{3\ \nearrow\text{outside}} \xrightarrow[\text{rule}]{\text{chain}} y' = \underbrace{3(f(x))^2}_{\substack{\text{derivative of}\\ \text{the outside cubic}\\ \text{function : } ▨^{3}}} \cdot \underbrace{f'(x)}_{\substack{\text{derivative of}\\ \text{the inside function}\\ f(x)}}$$

$$\xrightarrow[\text{given info}]{\text{sub } x=1 \text{, use the}} y'(1) = 3(f(1))^2 \cdot f'(1) = 3 \cdot 2^2 \cdot 1 \boxed{= 12}$$

(4) Find the tangent line to the graph of the function $g(x) = e^{f(x)}$ at $x = 0$ when $f(0) = 0$ and $f'(0) = -1$.

Recall : tangent line to a curve $y = f(x)$ at a given point $x = a$



⊛ We need two pieces of info to write equation of any line :

→ its slope and a point on the line .

⊛ What's special about tangent line ?

→ Its slope is equal to the derivative .

So $\boxed{m_{\tan} \text{ at } x = a = f'(a)}$

⊛ What's the point ?

→ $\boxed{\text{The tangency or touch point}}$ that is both on the line and on the graph , i.e. $(a, f(a))$

The tangent line equation at $x = a$

$$y - f(a) = f'(a)(x - a)$$

Or

$$y = f'(a)(x - a) + f(a)$$

Go back to the example (4):

To find m_{tan} at $x=0$ we need to find $g'(0)$.

$$\text{So} \quad g(x) = \underset{\text{out}}{e}^{\,f(x)\ \text{in}} \xrightarrow[\text{rule}]{\text{chain}} g'(x) = \underbrace{e^{f(x)}}_{\text{out}} \cdot \underbrace{f'(x)}_{\text{in}}$$

$$\xrightarrow[\text{at } x=0]{\text{evaluate}} g'(0) = e^{f(0)} \cdot f'(0)$$

$$\xrightarrow[\text{info}]{\text{use the}} g'(0) = \underbrace{e^{0}}_{1} \cdot (-1) = -1$$

$$\text{So} \quad \boxed{m_{tan} \text{ at } x=0 \text{ is } m=-1} \to f'(a)$$

Now find the tangency point: we have its x-coordinate $x=0$

plug into $g(x)$ to find the y-coordinate:

$$\xrightarrow{x=0} g(0) = e^{f(0)} = e^{0} = 1 \Rightarrow \boxed{(\underset{a}{0}, \underset{f(a)}{1}) \text{ the tangency point}}$$

$$\Rightarrow \quad y - 1 = -1(x-0)$$

$$\Rightarrow \quad \boxed{y = -x+1} \to \text{tangent line equation.}$$

Question 1: In the notation $y=f(x)$; what is the difference between x and y as variables?

⊛ x is the indepent variable taking arbitrary values in the domain however, the value of y depends on the value of x.

$\mapsto$ y is a function of x that obeys the rule f.

Question 2 : Compare the following expressions.

I: $y = \sqrt{4-x^2}$ and II: $y^2 + x^2 = 4$

I:

- y is isolated at one side of "=" with no operation on y ⤳ single y
- Another side of "=" contains only x with operations on x

→ $\boxed{y = f(x)}$ → Explicit form (a function)

II:

- x and y are combined both at the same side of the equation, with some operation on both x and y e.g. y^2 (NOT single y)
- Another side of the equation is a constant.

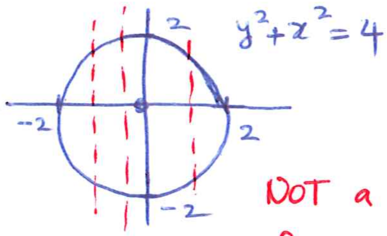
→ $\boxed{g(x,y) = \text{Const}}$ → Implicite form. (a curve in general)

Question 3 : Can you write II above as an explicit function? $y = f(x)$

⊛ To do so, we need to isolate y at one side and do some algebra to make the other side only in terms of x.

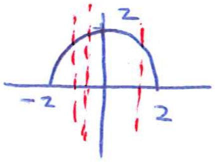
$$y^2 + x^2 = 4 \overset{\text{isolate}}{\Longrightarrow} y^2 = 4 - x^2 \xrightarrow[y]{\text{extract}} y = \pm\sqrt{4-x^2}$$

⊛ This is NOT a function, because for one x, we get two y-values. In fact, we check the graph of $y^2+x^2=4$, it's a circle with center at the origin and radius 2.



NOT a function

But $y = \sqrt{4-x^2}$ is the upper semi-circle with the same center and radius.



A function.