Question 4 : Does an implicit form exist that CAN be written explicitly ? Yes , many .

For example : $2y^3 + xy^3 = x$

$\xRightarrow{\text{isolate}} y^3(2+x) = x$

$\xRightarrow{\text{isolate}} y^3 = \dfrac{x}{2+x}$

$\Longrightarrow \boxed{y = \sqrt[3]{\dfrac{x}{2+x}}} \rightarrow y = f(x)$

THINK : What types of terms in an expression can help us predict whether the implicit form can be written explicitly or not ?

Lecture 2 , Jan 5

Goal : Differentiate the implicit forms .

Implicit Differentiation

Procedure : You are given an implicit curve to take its derivative.

You take the derivative of the terms in both sides of the equation, using the derivative rules that apply. But when you get to terms with y ; Keep in mind that y is implicitly a function of x ; in other words , there is a hidden $y = f(x)$.

So for example : $(y^2)' = (\underbrace{f(x)}_{\text{inside}}{}^{\overset{\text{out}}{2}})' = 2f(x) \cdot f'(x)$

$= 2yy'$.

or

$(xy)' \underset{\text{product rule}}{=} (1 \cdot y + x \cdot y')$

Example 1. Consider the curve

$$x^2 - 3xy + 7y = 5$$

a) Find $\frac{dy}{dx}$ by using implicit differentiation.

b) Find $\frac{dy}{dx}$ by $\underbrace{\text{writing the curve explicitly}}_{\text{means } y=f(x)}$ and then differentiating.

(a) To remember that y is a function of x we take $y=f(x)$ and replace y to rewrite the curve:

$$x^2 - 3x\,f(x) + 7f(x) = 5$$

Now take the derivative term by term in both sides of the equation

$$\underbrace{2x}_{\substack{\text{power rule}\\ (x^2)'}} - 3\underbrace{(\underbrace{1}_{(x)'}\cdot f(x) + x\,\underbrace{f'(x)}_{\substack{\text{derivative}\\ \text{of the second term}}})}_{\text{Product rule}} + 7f'(x) = \underbrace{0}_{\substack{(5)'\\ \text{derivative of}\\ \text{the constant}\\ \text{term.}}}$$

Go back to y:

$f(x) = y$

$f'(x) = y'$ $\Rightarrow$ $2x - 3y - 3xy' + 7y' = 0$

We're looking for y': isolate y' and solve it for y'

$$y'(-3x+7) = 2x - 3y \quad \Rightarrow \quad \underbrace{y'}_{\frac{dy}{dx}} = \frac{2x-3y}{-3x+7} \quad ✓$$

* Note: y' involves both x and y.

(b) First convert the curve to $y = f(x)$:

$$x^2 - 3x\underline{\underline{y}} + 7\underline{\underline{y}} = 5 \xrightarrow[y]{\text{isolate}} y(-3x+7) = 5 - x^2$$

$$\xrightarrow{\text{isolate}} y = \frac{5 - x^2}{-3x+7}$$

Now, we have the explicit form, we apply the rules from term 1:

$\longrightarrow$ Quotient rule:

<u>Exercise</u>: Use the quotient rule to find the derivative of y and verify that the two derivatives from part (a) and (b) are equal.

$\longrightarrow$ Now that we can find the derivative of implicit curves, we should be able to find the tangent line to an implicit curve at a given point as well.

$\longrightarrow$ As usual: $\left[m_{tan} \text{ at } x=a\right] = y'(a)$

and tangency point is $(a, f(a))$

But there might be a slight change $\rightsquigarrow$ next example.

Example 2. Let $\sin y + e^x y^2 = \sqrt{x} - e^y$

a) Find $\dfrac{dy}{dx}$.

(a) Remember that y is a function of x:

$$\underbrace{\underbrace{\sin}_{\text{out}}\ \underbrace{y}_{\text{in}}}_{\text{chain rule}} + e^x \underbrace{\underbrace{y}_{\text{in}}^{\overset{\text{out}}{2}}}_{\text{chain rule}} = \overset{x^{\frac{1}{2}}}{\sqrt{x}} - \underbrace{\underbrace{e}_{\text{out}}^{\overset{\text{in}}{y}}}_{\text{chain rule}}$$

take the derivative of each term:

$$\underbrace{\cos y}_{\text{outside derivative}} \cdot \underbrace{y'}_{\text{inside derivative}} + \underbrace{\left(e^x y^2 + \overbrace{2y}^{\text{outside derivative}} \cdot \underbrace{y'}_{\text{inside derivative}}\right)}_{\text{product rule}}$$

$$= \frac{1}{2} x^{\frac{1}{2}-1} - \underbrace{e^y}_{\text{outside derivative}} \cdot \underbrace{y'}_{\text{inside derivative}}$$

isolate all the terms involving y':

$$\cos y \cdot y' + 2yy' + e^y y' = \frac{1}{2} x^{-\frac{1}{2}} - e^x y^2$$

factor y': $$y'(\cos y + 2y + e^y) = \frac{1}{2\sqrt{x}} - e^x y^2$$

Solve for y': $$\frac{dy}{dx} = y' = \boxed{\frac{\frac{1}{2\sqrt{x}} - e^x y^2}{\cos y + 2y + e^y}}$$

→ Part (b) and (c): Next Page

(b) Find the slope of the tangent line at a point with $y = 0$?

means evaluate y' at the given point.

We have the derivative from part (a):

$$y' = \frac{\frac{1}{2\sqrt{x}} - e^x y^2}{\cos y + 2y + e^y}.$$

Unlike explicit functions, the derivative involves both x and y, so we need to have x and y coordinates of the tangency point to plug into y' and find the slope.

$\longrightarrow$ $y = 0$ what is x ? Use the equation of the curve:

$$\sin y + e^x y^2 = \sqrt{x} - e^y$$

$$\xrightarrow{y=0} \underbrace{\cancel{\sin 0}}_{0} + \underbrace{e^x \cdot \cancel{0}}_{0} = \sqrt{x} - \underbrace{\cancel{e^0}}_{1}$$

$$\longrightarrow \sqrt{x} = 1 \Rightarrow x = 1 \xRightarrow{\text{Point}} (1,0)$$

Now evaluate y' at $(1,0)$:

$$y' = \frac{\frac{1}{2\sqrt{1}} - \underbrace{\cancel{e^1 0}}_{0}}{\underbrace{\cancel{\cos 0}}_{1} + \underbrace{\cancel{2 \cdot 0}}_{0} + \underbrace{\cancel{e^0}}_{1}} = \frac{\frac{1}{2}}{2} = \frac{1}{4} = m_{tan}$$

(c) Find the equation of the tangent line at the point found in part (b).

We have all the required info: $m_{tan} = \frac{1}{4}$, point : $(1,0)$

$$y - 0 = \frac{1}{4}(x - 1) \Rightarrow \boxed{y = \frac{1}{4}x - \frac{1}{4}}$$