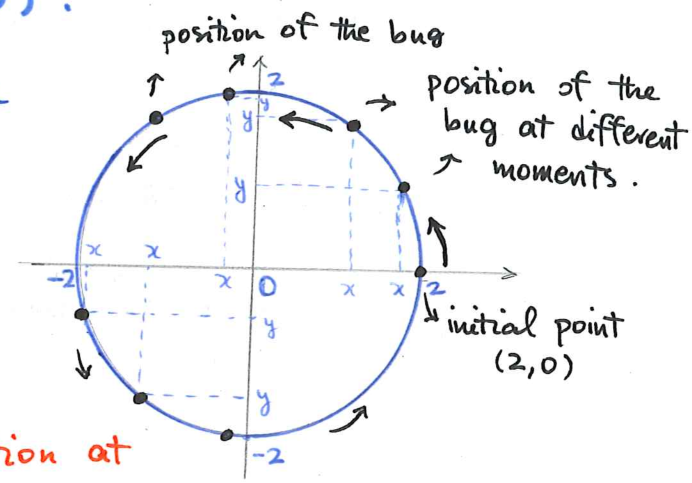
Lecture 4, 5   Jan 10, 12

In our last classes, we learned that a circle can be seen as an implicit curve since in its expression y is not isolated at one side. For example, a circle with centre at the origin and radius 2 has the equation $x^2 + y^2 = 4$.

Now, we consider the following scenario:

→ A bug is moving on the circle $x^2 + y^2 = 4$ and it's initially at the point $(2,0)$.

If we record the bug's position at different moments then:



Question 1: What is the new variable entered in this scenario?

time : t. We record the position at different moments.

Question 2: What are the quantities that are changing over time?

Since the position is changing, therefore, x and y coordinates are changing.

As you see at each time t, x and y have different values, this means that x and y are functions of time.

A New Independent Variable : time (t)

$\xrightarrow{\text{functions of } t}$ $x(t)$ and $y(t)$

Now let's compare the two cases for the same curve :
A circle with centre at the origin and radius r

$$x^2 + y^2 = r^2$$

Case 1 : Without time

x ⇝ independent variable

y ⇝ depending on x : $y(x)$ or $f(x)$

⇝ y is implicitly a function of x

⇝ derivative is with respect to x

⇝ $\frac{dy}{dx}$ or $y'(x)$

Case 2 : With time

t ⇝ independent variable

x & y ⇝ both depend on time

⇝ x & y are implicitly functions of t

⇝ $x(t)$ & $y(t)$

⇝ derivative with respect to t

$\frac{dy}{dt}$ or $y'(t)$

$\frac{dx}{dt}$ or $x'(t)$

Let's differentiate both sides :

with respect to x

$$(x^2 + \underbrace{y^2}_{(f(x))^2})' = (r^2)'$$

a constant number

$$2x + 2y \cdot y' = 0$$

with respect to t

$$(\underbrace{x^2}_{(x(t))^2} + \underbrace{y^2}_{(y(t))^2})' = (r^2)'$$

still a constant

$$2x \cdot x' + 2y \cdot y' = 0$$

They are two different derivatives with respect to different variable.

<u>Note</u> : When taking derivatives in general, we should pay attention to the variable with respect to which we differentiate.

<u>Question</u> : What's derivative of x with respect to x ? i.e. $\frac{dx}{dx}$?

In term 1; you learned that a physical interpretation of the derivative is "Rate of Change". We are going to solve some word problems that deal with rate of change in some physical quantity with respect to time.

Related Rates.

Not a new calculus topic, we need to learn strategies to translate word problems into mathematical language and use calculus to solve them.

→ **Setting:** A (physical) situation is given and it describes the change in some quantities over time. For example; Change in distance, length, area, volume, angle, radius, perimeter, ...

→ Some information about the value of the quantities at one instant in time is given.

→ **Question:** If two or more quantities are related to each other by some mathematical formulas, then find the rate of change (increase/decrease) in one of them, assuming we know the rate of change of the others.

→ **Strategy** : Follow the steps below :

1. Draw a diagram or a picture of the situation.

2. Read the problem carefully, determine the quantities that are changing, those are the variables. Assign some letter to the variables.

   Note : Not all letters (A, B, d, e ...) are variables. Sometimes, a fixed quantity can be denoted with a letter too. Be careful about these fixed quantities when differentiating.

3. Set up an equation that relates the variables.

   Note : Check your diagram carefully and see what geometric formulas can be applied. We usually apply one of the following :

   a) Pythagoras
   b) Similar triangles
   c) Formulas for area and volume.
   d) Trigonometric ratios.

4. Differentiate both sides of the equation that you've found with respect to time (t).

5. Use the given information in the question to substitute the known terms and solve for the unknown quantity.

   Note : The unknown quantity is often the rate of change in one of the variables. For example, $y'(t)$, $A'(t)$, $r'(t)$, ...