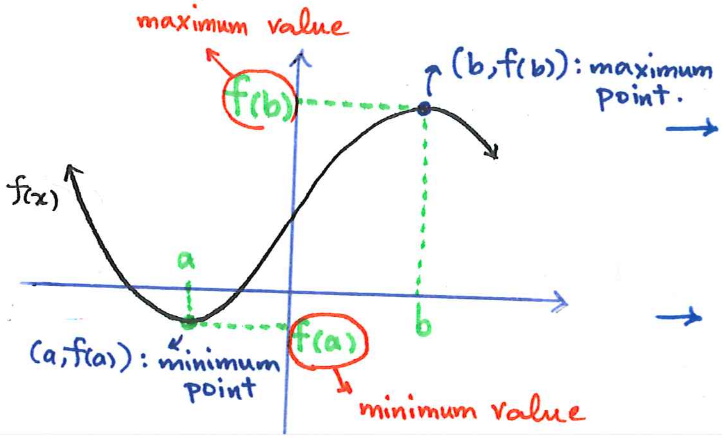
Lecture 9, Jan 22

In the past lectures, we learned about related rates as one application of the derivative to be the rate of change in some physical quantity. Now, we would like to learn how we can use the derivative of a function (f'), to gain information about the function itself and its graph. In the next couple of lectures, we focus on developing steps to graph complicated functions.



→ What can we say about the points $(a, f(a))$ and $(b, f(b))$?

→ What's special about these two points?

The point $(a, f(a))$ is the lowest point on the graph. (smallest y-value)

↳ $(a, f(a))$ is a minimum point ⇝ $f(a)$ min value

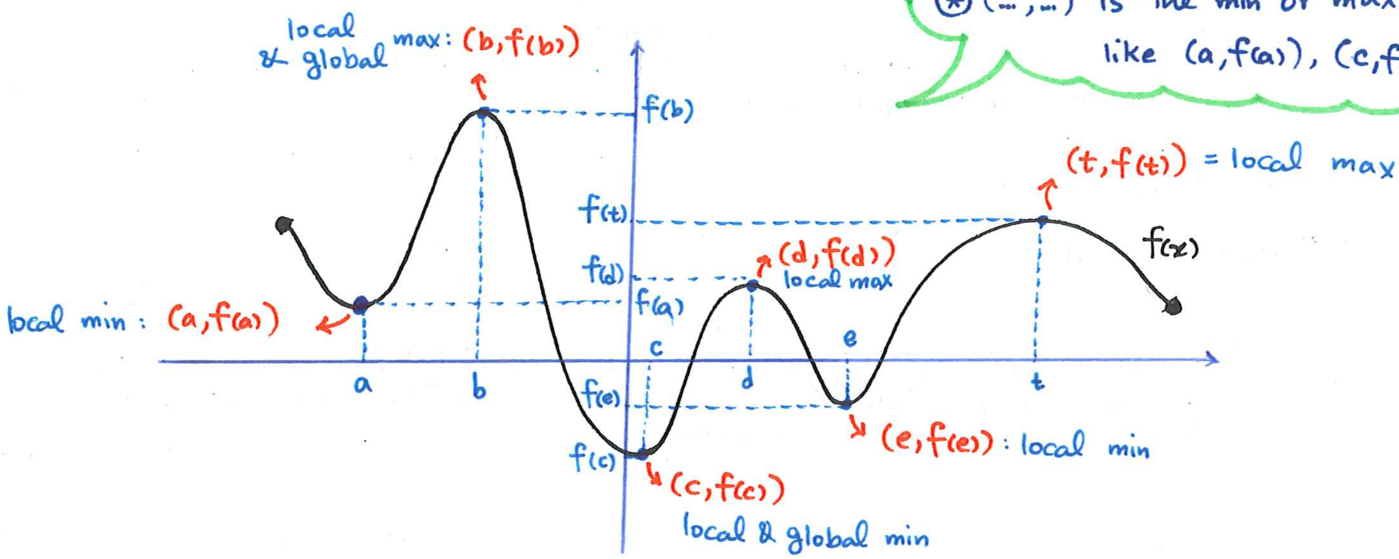
The point $(b, f(b))$ is the highest point on the graph. (largest y-value)

↳ $(b, f(b))$ is a maximum point ⇝ $f(b)$ max value

⊛ Note that if we continue the graph from left or right we may get different smallest/largest values, but for now we don't worry about the function at the end-points.

Consider the following function:

> ⊛ y-values are min or max values, like $f(a)$, $f(c)$ ...
>
> ⊛ $(\ldots,\ldots)$ is the min or max point. like $(a,f(a))$, $(c,f(c))$, ...



→ What's the highest (maximum) and lowest (minimum) points of $f(x)$?

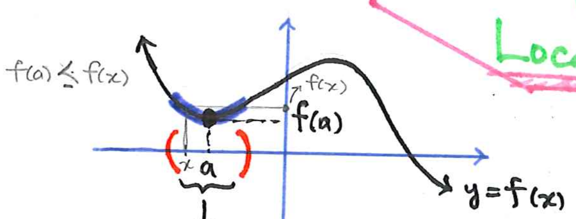
$(b, f(b))$ is the highest ⇝ max , $(c, f(c))$ is the lowest ⇝ min

→ Are the points $(a, f(a))$, $(d, f(d))$, $(e, f(e))$ and $(t, f(t))$ are max ~~or~~ min in some sense?

↳ In the sense that: we just restrict our search for max or min in a neighborhood and NOT on the whole domain?

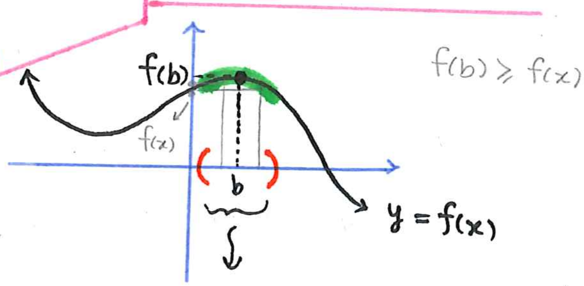
locally

Local Maximum / Minimum
or
Local Extremum



a neighborhood around a

Check graph: in this neighborhood $f(a)$ is the lowest y-value.



a neighborhood around b

Check the graph: $f(b)$ is the highest y-value in this neighborhood.

Translate these into math terminology.

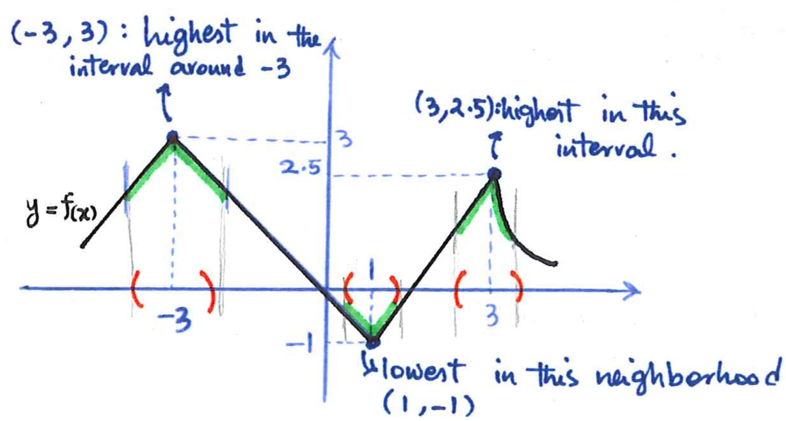
Definition 1. (Local Minimum) A function f has a local minimum at "a" if $f(a) \leq f(x)$ for all x's in some interval around "a".

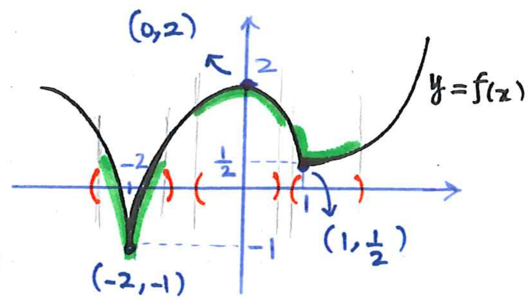
Definition 2. (Local Maximum) A function f has a local maximum at "b" if $f(b) \geq f(x)$ for all x's in some interval around "b"

Note 1. When finding local extrema of a function f, we only care about the y-values at some neighborhood NOT the whole domain

Note 2. There are global min/max for a function that are max and min on the whole domain of the function. We ignore them for now. Later in the course we deal with them.

Let's see some more examples. Find the local extrema.





local min

(1, -1)

x-value at which y-value is the minimum

y-value is the minimum value

local max

(-3, 3)

(3, 2.5)

x-val

y-values are maximum values in two different neighborhoods.

local min

(-2, -1)

$(1, \frac{1}{2})$

x-values corresponding to min-values

local max

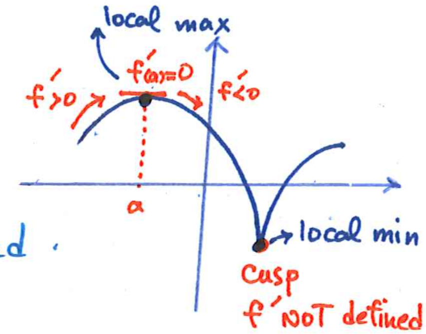
x-values

(0, 2)

y-value for extrema locally

Important Question 1. Given a local extremum (max or min) point, what information can be extracted about the derivative?

→ If we check the graph of the examples we saw, we observe that at the x-coordinate of a local extremum:



either slope of the tangent line $=0 \rightsquigarrow f'(a)=0$

or slope is NOT defined $\rightsquigarrow f'(a)$ NOT defined.

Important Question 2. Is the converse true? i.e. If we know that at a given point, the derivative is 0 or NOT defined, is that point a local extremum? Not necessarily, Not all such points are local extrema, but they are the possible

→ candidates for the extrema.

Definition 3. **Critical Numbers**

A critical number for a function f is a value "$x=a$" in the domain of f if

either $f'(a)=0$
or $f'(a)$ is NOT defined.

Recall: Graphically, a function is NOT differentiable at corners and cusps.

So by the definition, NO derivative at corners & cusps $\rightsquigarrow$ They are critical numbers.

In the expression of the derivative there is a
- fraction
- square root
- log function

$\rightsquigarrow$ problematic functions

Therefore, to find the critical numbers to a given function the first step is to find the derivative. Then we find the points at which f' is 0 or NOT defined.

Next week, we'll see how to find which of the critical points (our candidates) are local max or min.

Examples. Find the critical numbers of the following functions.

1) $f(x) = x^3 - 6x^2 + 5$

$f'(x) = 3x^2 - 12x$

Where $f' = 0$?

Where f' NOT defined? $3x^2 - 12x$ is defined everywhere; this case NEVER happens.

$f' = 0 \Rightarrow 3x^2 - 12x = 0$

$\xrightarrow[\text{the equation}]{\text{solve}} 3x(x-4) = 0$

$3x = 0 \Rightarrow x = 0$

$x - 4 = 0 \Rightarrow x = 4$

critical numbers.

What are the critical $\underline{\text{Points}}$? find the y-coordinates ⇝ plug into f.

$(a, f(a))$

$f(0) = 0 - 0 + 5 = 5 \rightsquigarrow (0, 5)$

$f(4) = 4^3 - 6\cdot(4)^2 + 5 = -27 \rightsquigarrow (4, -27)$

critical points.

2) $f(x) = x^4 - 4x^2 + 10$

$f'(x) = 4x^3 - 8x$

Is always defined ✓

$\longrightarrow f' = 0$

$4x^3 - 8x = 0$

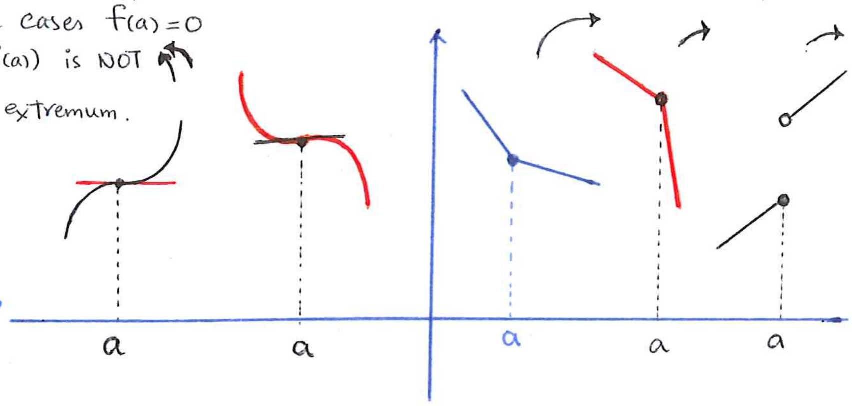
$4x(x^2 - 2) = 0$

$4x = 0 \Rightarrow \boxed{x = 0}$

$x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow \boxed{\begin{array}{l} x = \sqrt{2} \\ x = -\sqrt{2} \end{array}}$

→ Recall important question 2 in the last lecture, we know that critical numbers only give the possible locations of local extrem[a] and some critical numbers are NOT local extrema. We give preci[se] details on this next week, but graphically the following graph[s] show critical numbers that are NOT local min or max.

In both cases $f'(a) = 0$ but $(a, f(a))$ is NOT a local extremum.



In all these three cases, the derivative is NOT defin[ed] at "a" (there's a corne[r] or a discontinuity), bu[t] again $(a, f(a))$ is neith[er] a local min nor a loc[al] max.

In all the graphs above, $x = a$ is a critical number since either $f'(a) = 0$ or $f'(a)$ is NOT defined

But

the point $(a, f(a))$ is NOT a local extremum.

Test Yourself. True/False?

1) If the function f has a critical number at "c", then f has a local extremum at $x = c$.

2) The function g is differentiable everywhere and it has a local minimum at $x = 1$, then $f'(1) = 0$

3) $f(x) = |x|$ has a local min at $x = 0$

4) Not all critical points are local extrema.

5) All local extrema are critical points.