Lecture 10 . Jan 24

More examples. Find the critical numbers for the following function

1) $f(x) = \sqrt[3]{x}$

2) $f(x) = (x-1)^2(x-3)$

3) $g(t) = \dfrac{t^2+1}{t^2-t-6}$

4) $h(x) = 10x\,e^{3-x^2}$

5) $g(r) = r e^{-5r}$

6) $h(\theta) = \sin\theta - \theta$ in $[0, 2\pi]$

7) $f(x) = x\ln x - 2x$

8) $g(x) = \sqrt[3]{x^2}\,(2x-1)$

Solutions .

1) $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}\,x^{\frac{1}{3}-1} = \frac{1}{3}\,x^{-\frac{2}{3}} = \frac{1}{3}\,\frac{1}{x^{2/3}}$

in computations, always make a negative power, positive by forming a fraction .

$\Rightarrow f'(x) = \dfrac{1}{3x^{\frac{2}{3}}}$ two cases

$\longrightarrow f'(x) = 0 \xrightarrow{\text{when}}$ numerator $= 0 \leadsto$ NEVE

$\longrightarrow f'(x)$ NOT defined $\xrightarrow[\text{when}]{}$ denominator $= 0$

f' is NOT defined when

$3x^{\frac{2}{3}} = 0 \Rightarrow x^{\frac{2}{3}} = 0 \Rightarrow \boxed{x = 0}$ — the only critical number

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2) $f(x) = (x-1)^2(x-3)$

product rule : $f'(x) = 2(x-1)(x-3) + (x-1)^2$

$\xrightarrow[\text{factor } (x-1)]{\text{simplify}} = (x-1)\left[2(x-3) + (x-1)\right]$

$= (x-1)(2x-6+x-1)$

$\Rightarrow f'(x) = (x-1)(3x-7)$

$\longrightarrow f'$ NOT defined $\rightsquigarrow$ NEVER happens

$\longrightarrow f'(x)=0 \rightsquigarrow$ solve an equation

$f'(x)=0 \Rightarrow (x-1)(3x-7)=0 \Rightarrow \begin{cases} x-1=0 \Rightarrow x=1 \\ 3x-7=0 \Rightarrow x=\frac{7}{3} \end{cases}$ Critical numbers

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3) $g(t) = \dfrac{t^2+1}{t^2-t-6}$

Quotient Rule $\rightsquigarrow$ $g'(t) = \dfrac{2t(t^2-t-6)-(2t-1)(t^2+1)}{(t^2-t-6)^2}$ $(2t^3+2t-t^2-1)$

$$= \frac{\cancel{2t^3}-2t^2-12t-\cancel{2t^3}-2t+t^2+1}{(t^2-t-6)^2}$$

$$g'(t) = \frac{-t^2-14t+1}{(t^2-t-6)^2}$$

$g'(t)=0$

when $-t^2-14t+1=0$

or $t^2+14t-1=0$

$\downarrow$

We can NOT factor, so we need to use the quadratic formula:

Recall: $ax^2+bx+c=0$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$g'(t)$ NOT defined when

$t^2-t-6=0$

$\downarrow$

If this case happens, then the original function $g(t)$ is NOT defined either, so we don't need to consider this case

Make sure the critical numbers that you find belong to the domain as well $\overset{\text{i.e.}}{\rightsquigarrow}$ the function itself must be defined at those points.

$$t^2 + 14t - 1 = 0 \quad \Rightarrow \quad t = \frac{-14 \pm \sqrt{14^2 - 4\cdot 1\cdot(-1)}}{2}$$

$$\Rightarrow \quad \boxed{t = \frac{-14 \pm \sqrt{200}}{2}}$$ — two critical numbers

you can simplify:

$$\sqrt{200} = \sqrt{100 \times 2} = 10\sqrt{2} \quad \Rightarrow \quad t = \frac{-14 \pm 10\sqrt{2}}{2} = \frac{-7 \pm 5\sqrt{2}}{1}$$

$\rightarrow t_1 = -7 + 5\sqrt{2}$

$\rightarrow t_2 = -7 - 5\sqrt{2}$

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4) $h(x) = 10\, x\, e^{3-x^2}$

Product Rule & Chain rule $\rightsquigarrow$ $h'(x) = 10\left(e^{3-x^2} + x \cdot e^{3-x^2} \cdot (-2x)\right)$

$$h'(x) = 10\left(e^{3-x^2} - 2x^2 e^{3-x^2}\right)$$

$h'(x) = 0$

$\Downarrow$

$$\underbrace{e^{3-x^2}}_{} - 2x^2 \underbrace{e^{3-x^2}}_{} = 0$$

common factor

$$e^{3-x^2}\left(1 - 2x^2\right) = 0$$

$h'(x)$ NOT defined

exponential function and x^2 are defined everywh

This case NEVER happen

$e^{3-x^2} = 0 \rightsquigarrow$ NEVER happens $\rightsquigarrow$ exp function always POSITIVE.

$\Rightarrow$

$1 - 2x^2 = 0 \rightsquigarrow 1 = 2x^2 \rightarrow x^2 = \frac{1}{2} \Rightarrow \underbrace{x = \frac{1}{\sqrt{2}},\ x = -\frac{1}{\sqrt{2}}}_{\text{two critical nu}}$

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5) $g(r) = r e^{-5r}$

$g'(r) = e^{-5r} + r e^{-5r} \cdot (-5) = e^{-5r} - 5r e^{-5r}$

product rule

$g'(r) = e^{-5r}(1-5r)$ → $g'(r)$ is defined everywhere

↓

$g'(r) = 0$

⇓

$e^{-5r}(1-5r) = 0$

→ $e^{-5r} = 0$ → NEVER happens

→ $1 - 5r = 0$ → $\boxed{r = \frac{1}{5}}$ — critical number

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6) $h(\theta) = \sin\theta - \theta$ in $[0, 2\pi]$

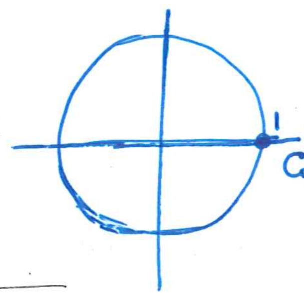
$h'(\theta) = \cos\theta - 1$

→ h' NOT defined → $\cos\theta$ is defined everywhere → NEVER happens

→ $h'(\theta) = 0$

$\Rightarrow \cos\theta - 1 = 0 \Rightarrow \cos\theta = 1$ → for what angle in $[0, 2\pi]$, Cos is equal to 1

$\cos\theta = 1 \Rightarrow \underbrace{\theta = 0,\ \theta = 2\pi}_{\text{critical numbers.}}$



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7) $f(x) = x\ln x - 2x$

$f'(x) = \ln x + x \cdot \frac{1}{x} - 2 = \ln x + 1 - 2$

$f'(x) = \ln x - 1$ ⟶ f' is NOT defined when $x = 0$ because $\ln 0$ is undefined BUT $x = 0$ is NOT in the domain either, so we ignore it.

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$f'(x) = 0$

⇓

$\ln x = 1$

⇓

$e^{\ln x} = e^{1} \Rightarrow \boxed{x = e}$ — critical number

8) $g(x) = x^{\frac{2}{3}}(2x-1)$ $\rightsquigarrow$ easier to distribute first.

$= 2x^{\frac{2}{3}} \cdot x - x^{\frac{2}{3}}$ $\qquad x^m \cdot x^n = x^{m+n}$

$= 2x^{\frac{5}{3}} - x^{\frac{2}{3}}$ $\qquad x^{\frac{2}{3}} \cdot x = x^{\frac{2}{3}+1}$

$g'(x) = 2 \cdot \frac{5}{3} x^{\frac{5}{3}-1} - \frac{2}{3} x^{\frac{2}{3}-1}$

$g'(x) = \frac{10}{3} x^{\frac{2}{3}} - \frac{2}{3} x^{-\frac{1}{3}}$ $\rightarrow$ make it positive

$= \frac{10}{3} x^{\frac{2}{3}} - \frac{2}{3} \frac{1}{x^{\frac{1}{3}}}$ $\qquad$ common denominator : $3x^{\frac{1}{3}}$

$= \dfrac{10x^{\frac{2}{3}} \cdot x^{\frac{1}{3}} - 2}{3x^{\frac{1}{3}}}$ $\qquad$ * $x^{\frac{2}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{2}{3}+\frac{1}{3}}$

$g'(x) = \dfrac{10x - 2}{3x^{1/3}}$

$\rightarrow$ $g'(x)=0$ $\rightsquigarrow$ numerator $= 0$

$\rightarrow$ g' NOT defined $\rightsquigarrow$ denominator $= 0$

$10x - 2 = 0$ $\qquad 3x^{1/3} = 0$

$10x = 2$ $\qquad \boxed{x = 0}$ $\rightarrow$ Note that $x=0$ is in the domain of $g(x)$. In fact $g(0)=0$ But it's NOT in the domain of g' so it is a critical number.

$\Rightarrow \boxed{x = \frac{2}{10} = \frac{1}{5}}$

critical numbers